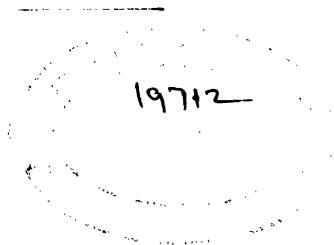# A TEXT - BOOK

OF

APPLIED MECHANICS

AND

MECHANICAL ENGINEERING.

IN FIVE VOLUMES.

VOLUME II.—STRENGTH OF MATERIALS.

ELEVENTH EDITION.

1971

WICKSTEED'S FIFTY-TON VERTICAL TESTING MACHINE AS MADE BY JOSHUA BUCKTON & CO. LEEDS.

For Description see Prof. Jamieson's "Text-Book on Applied Mechanics" in the Lecture on "Testing Machines."

A TEXT-BOOK

OF

APPLIED MECHANICS

AND MECHANICAL ENGINEERING.

Specially Arranged
For the Use of Engineers Qualifying for the Institution of Civil Engineers
The Institution of Mechanical Engineers, The Institution of Electrical
Engineers, The Diplomas and Degrees of Technical Colleges and
Universities, and Honours Certificates of the City and Guilds
of London Institute, in Mechanical Engineering,
and for Engineers Generally.

BY

ANDREW JAMIESON, M.Inst.C.E.,

FORMERLY PROFESSOR OF ENGINEERING IN THE GLASGOW ROYAL
TECHNICAL COLLEGE.

VOLUME II.—STRENGTH OF MATERIALS.

REVISED BY

EWART S. ANDREWS, B.Sc.Eng.(Lond.),

LECTURER IN THE ENGINEERING DEPARTMENT OF GOLDSMITHS' COLLEGE, NEW CROSS; FORMERLY
DEMONSTRATOR IN THE ENGINEERING DEPARTMENT OF UNIVERSITY COLLEGE, LONDON.

With Numerous Diagrams, Special Plates, and Examination Questions.

ELEVENTH EDITION.

(Second Impression.)

\197\2

LONDON:

CHARLES GRIFFIN & COMPANY, LIMITED,

42 DRURY LANE, W.C. 2.

1946

▼
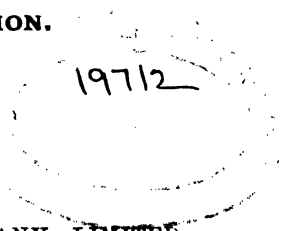
PREFACE

TO THE NINTH EDITION.

In editing the present Revised Edition of Professor Jamieson's "Applied Mechanics"—Vol. II., *Strength of Materials*—great care has been taken to preserve the treatment and method of presentation which have characterised Professor Jamieson's work, and have proved of great assistance to a very wide circle of readers.

Additional subject-matter has been introduced, dealing principally with the behaviour of various materials under test, the Lamé theory of thick cylinders, the Brinell test for hardness, and the properties of British Standard Sections.

The opportunity has been taken in the revision of incorporating in the normal place certain portions which previously appeared in Appendices.

The most recent papers for the Associate Membership Examination of the Institution of Civil Engineers have been incorporated by the kind permission of the Council.

INSTRUCTIONS FOR ANSWERING HOME EXERCISES.

----

1. Use ordinary foolscap paper, and write on the left side *only*, leaving the facing page blank for corrections and remarks.

2 Put the date of the Exercises at the left-hand top corner; your Name and Address in full, the name of the Subject or Section, as well as number of Lecture or Exercise, in the centre of the first page. The number of each page should be put in the right-hand top corner.

3. Leave a margin 1¼ inches wide on the left-hand side of each page, and in this margin place the *number* of the question *and nothing more*. Also, leave a clear space of *at least* 2 inches deep between your answers.

4. Be sure you understand *exactly* what the question requires you to answer, then give *all* it requires, but *no more*. If unable to fully answer any question, write down your own best attempt and state your difficulties.

5. Make your answers concise, clear, and exact, and always accompany them, if possible, by an *illustrative sketch*. Try to give (1) Side View, (2) Plan, (3) End View. Where asked, or advisable, give Sections, or Half Outside Views and Half-Sections for (1), (2), and (3).

6. Make all sketches large, open, and in the centre of the page. Do not crowd any writing about them. Simply print sizes and index letters (or names of parts), with a bold Sub-heading of what each figure or set of figures represent.

7. The character of the sketches will be carefully considered in awarding marks to the several answers. Neat sketches and "*index letters*," having the first letter of the name of the part, will always receive more marks than a bare written description.

8. Reasonable and easily intelligible contractions (*e.g.*, mathematical, mechanical or electrical, and chemical symbols) are permitted.

CONTENTS.

LECTURE VI.

Strength and Elasticity of Materials. Testing and Testing Machines.

APPENDIX.

CONTENTS OF VOLUMES.

MECHANICAL ENGINEERING SYMBOLS, ABBREVIATIONS, AND INDEX LETTERS

USED IN VOLUMES I. TO V.

OF PROFESSOR JAMIESON'S "APPLIED MECHANICS."

Prefatory Note.—It is very tantalising, as well as a great inconvenience to Students and Engineers, to find so many different symbol letters and terms used for denoting one and the same thing by various writers on mechanics. It is a pity, that British Civil and Mechanical Engineers have not as yet *standardised* their symbols and nomenclature as Chemists and Electrical Engineers have done. The Committee on Notation of the Chamber of Delegates to the International Electrical Congress, which met at Chicago in 1893, recommended a set of "Symbols for Physical Quantities and Abbreviations for Units," which have ever since been (almost) universally adopted throughout the world by Electricians.* This at once enables the results of certain new or corroborative investigations and formulæ, which may have been made and printed anywhere, to be clearly understood anywhere else, without having to specially interpret the precise meaning of each symbol letter.

In the following list of symbols, abbreviations and index letters, the *first* letter of the chief noun or most important word has been used to indicate the same. Where it appeared necessary, the *first* letter or letters of the adjectival substantive or qualifying words have been added, either as a following or as a subscript or suffix letter or letters. For certain specific quantities, ratios, coefficients and angles, small Greek letters have been used, and I have added to this list the complete Greek alphabet, since it may be refreshing to the memory of some to again see and read the names of these letters, which were no doubt quite familiar to them when at school.

* These "Symbols for Physical Quantities and Abbreviations for Units" will be found printed *in full* in the form of a table at the commencement of Munro and Jamieson's *Pocket-Book of Electrical Rules and Tables.* If a similar recommendation were authorised by a committee composed of delegates from the chief Engineering Institutions, it would be gladly adopted by "The Profession" in the same way that the present work of "The Engineering Standards Committee" is being accepted.

TABLE OF MECHANICAL ENGINEERING QUANTITIES, SYMBOLS, UNITS AND THEIR ABBREVIATIONS.

(As used in Vols. I. and II. of Prof. Jamieson's "Applied Mechanics.")

Quantities.	Symbols.	Defining Equations.	Practical Units.	Abbreviations of the Practical Units.
FUNDAMENTAL.				
Length, . . .	L, l	...	Yard, . . .	yd.
			Foot, . . .	ft.
			Inch, . . .	in.
Mass, . . .	M, m	...	Pound, . . .	lb.
Time, . . .	T, t	...	Second, . .	s.
			Minute, . .	m.
			Hour, . . .	h.
GEOMETRIC.				
Surface, . . .	S, s	$S = L^2$	Square foot, . .	sq. ft.
			Square inch, . .	sq. in.
Volume, . . .	V	$V = L^3$	Cubic foot, . .	cb. ft.
			Cubic inch, . .	cb. in.
Angle, ∠ . .	$\left\{ \begin{array}{c} a, \beta \\ \theta, \phi \end{array} \right\}$	$a = \dfrac{\text{arc}}{\text{radius}}$	Degree, . .	1°
			Minute, . .	1′
			Second, . .	1″
			Radian $= \dfrac{180°}{\pi}$.	rn.
MECHANICAL.				
Velocity, . .	v	$v = \dfrac{L}{T}$	Foot per second, .	$\dfrac{\text{ft.}}{\text{s.}}$
Angular velocity, .	ω	$\omega = \dfrac{v}{L} = \dfrac{\theta}{t}$	Revs. per second, .	r.p.s.
			Revs. per minute,	r.p.m.
			Radians per second,	ω
Acceleration, . .	a, g	$a = \dfrac{v}{T}$	Foot per sec. per sec.	$\dfrac{\text{ft.}}{\text{s}^2}$
Force, . . .	F, f	...	Pound weight (gravitational unit),	lb. wt. (or lb.)
	W, w	$F = M a$	Poundal (absolute unit),	pdl.
Pressure (per unit area),	p	$p = \dfrac{F}{s}$	Pound per sq. inch,	lb. □″
Work, . . .	$(W h)$	$W h = F L$	Foot-pound, . .	ft.-lb.
Potential energy, .	E_P	$E_P = W h$	Foot-pound, . .	ft.-lb.
Kinetic energy, .	E_K	$E_K = \dfrac{W v^2}{2 g}$	Foot-pound, . .	ft.-lb.
Power or activity, .	HP	$\text{H.P.} = \dfrac{W h}{T}$	Horse power, .	H.P.
			Ft.-lb. per min., .	ft.-lb./m.
			Ft.-lb. per sec., .	ft.-lb./s.
Moment of inertia, .	I	$I = M k^2$	... ...	lb.-ft.²
Density, . . .	ρ	$\rho = \dfrac{M}{V}$	Pound per cb. ft., .	$\dfrac{\text{lb.}}{\text{ft.}^3}$
			Pound per cb. in., .	$\dfrac{\text{lb.}}{\text{in.}^3}$

OTHER SYMBOLS AND ABBREVIATIONS IN VOLS. I. AND II.

A for Areas.
B, b ,, Breadths.
C, c, k ,, Constants, ratios.
c.g. ,, Centre of gravity.
D, d ,, Diameters depths, deflections.
D_1, D_2, D_3 ,, Drivers in gearing.
E ,, Modulus of elasticity.
e ,, Velocity ratio in wheel gearing.
F_1, F_2, F_3 ,, Followers in gearing.
f_s, f_t ,, Forces of shear and tension.
H, h ,, Heights, heads.
H.P., h.p. ,, Horse-power.
B.H.P. ,, Brake horse-power.
E.H.P. ,, Effective ,,
I.H.P. ,, Indicated ,,
k ,, { Radius of gyration, or, Coef. of discharge in hydraulics.
N, n ,, Numbers—*e.g.*, number of revs. per min., number of teeth, &c.
P, Q ,, Push or pull forces.
R_1 R_2 ,, Reactions, resultants, radii, resistances.
s ,, { Seconds, space, surface. Displacement, distance.
S F ., Shearing force.
T M ,, Torsional moment.
T R ,, Torsional resistance.
B M ,, Bending moment.
M R ,, Moment of resistance.
R M ,, Resisting moment.
T_d, T_s ,, Tensions on driving and slack sides of belts or ropes, &c.
W_L, W_T, W_U ,, Lost, total, and useful work.

x, y, z for Unknown quantities.
Z ,, Modulus of section.
Z_t ,, ,, tension.
Z_c ,, ,, compression.

Δ, δ, d for Differential signs which are prefixed to another letter; then the two together represent a very small quantity.
ϵ, e ,, Represents base of Naperian Logs $= 2\cdot7182$; for example, $\log_\epsilon 3 = 1\cdot1$.
η ,, Efficiency.
λ ,, Length ratio of ship to model.
μ ,, Coefficient of friction.
π ,, Circumference of a circle $\div$ its diameter.
ρ ,, Radius of curvature, radian.

Σ for Symbol for sum total of a number of quantities.
$\int_0^x$,, Sign of integration or summation between limits 0 and x.
$\sim$,, Sign for the difference between two quantities.
$\square$,, Sign for square—*e.g.*, 10 $\square'' =$ 10 square inches.
$-$,, Sign over two letters, $\overline{PQ}$ for a force acting from P to $\rightarrow$ Q, means that they represent a vector quantity, which has (1) magnitude, (2) direction, (3) sense.
$\geqslant$,, Sign for equal to or greater than.
$\leqslant$,, Sign for equal to or less than.

GREEK ALPHABET.

A	α	Alpha.	I	ι	Iota.	P	ρ	Rho.
B	β	Beta.	K	κ	Kappa.	Σ	σ or ς	Sigma.
Γ	γ	Gamma.	Λ	λ	Lambda	T	τ	Tau.
Δ	δ	Delta.	M	μ	Mu	Υ	υ	Upsilon.
E	ϵ	Epsilon.	N	ν	Nu	Φ	ϕ	Phi.
Z	ζ	Zeta.	Ξ	ξ	Xi.	X	χ	Chi.
H	η	Eta.	O	o	Omicron.	Ψ	ψ	Psi.
Θ	θ	Theta.	Π	π	Pi.	Ω	ω	Omĕga.

VOLUME II.

ON

STRENGTH OF MATERIALS.—STRESS, STRAIN, ELASTICITY, FACTORS OF SAFETY, RESILIENCE, CYLINDERS, CHAINS, SHAFTS, BEAMS, AND GIRDERS.

LECTURE I.

Stress.—When a piece of material is subjected to the action of external forces they tend to cause the material to change its shape or form. The particular way in which the change takes place depends upon the manner in which the load is applied. This tendency gives rise to certain forces within the material which offer resistance to the change. These internal forces are generally called *stresses;* but the term *Stress* which we have now to consider has a somewhat more definite meaning. By the principle of the equality of action and reaction, we know that so long as no rupture of the material takes place, the algebraic sum of the components of the internal forces in the direction of the load at any section of the material must be equal to the load. This principle enables us to express the *internal* in terms of the *external* forces. It is a fundamental fact that, for a given load, the amount of resistance to be contributed by each individual fibre or part composing a section will be less or greater, according as the number of such fibres or parts is greater or less; or as we usually regard it, according as there is more or less area of section. This introduces us to the conception of *distributed* force, and paves the way towards gaining definite and clear ideas regarding the strength of materials.

DEFINITION.—**Intensity of stress is the resistance or reaction to strain per unit area of section.** For brevity it is usually ca'led the Stress. Stresses may be of three different kinds, depending on the direction of the applied force with reference to the section on which the stress is estimated.

(1) If the applied force is normal or at right angles to the section, and acting *away* from it, the stress is called *tensile.*

2

(2) If acting *towards* the section, the stress is termed *compressive*

(3) If the direction of the applied force be *parallel* to the section, then the stress is named a *shearing* stress.

It is evident that if the applied force be acting in a direction inclined to the given section, it will cause both a shearing and a direct stress, the latter being tensile or compressive, according as the force is directed away from or towards the section.

When the applied force acts in such a way that we know that its effect is uniformly distributed over the section we are considering, then we estimate the stresses as follows :—

Let P_n = The applied load (or its component) acting normally to the section in lbs. or tons.

„ A = The area of the section (usually in square inches).

„ f = The direct stress, which may be either tensile or compressive.

„ P_t = The applied load (or its component) acting tangentially to the section in lbs. or tons.

„ f_s = The shearing stress.
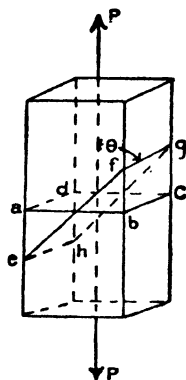
ILLUSTRATING NORMAL AND TANGENTIAL STRESSES.

Then,
$$f = \frac{P_n}{A}.$$

And,
$$f_s = \frac{P_t}{A}.$$
$$\left. \right\} \quad \cdots \cdots \cdots \quad \text{(I)}$$

Relation between Normal and Tangential Stresses.—Let *a b c d* be the section of a bar normal to the direction of the applied force P, and *e f g h* another section making an angle θ, with the direction of P; and let the area of *a b c d* be A square inches.

Thus, the stress on *a b c d* is :—
$$f = \frac{P}{A};$$

But, on the area *e f g h*, we have a normal force :—
$$P_n = P \sin \theta,$$

And a tangential force :—
$$P_t = P \cos \theta.$$

Now, the area *e f g h* = $\dfrac{\text{area } a\,b\,c\,d}{\sin \theta} = \dfrac{A}{\sin \theta}$

If f_n and f_t be the normal and tangential stresses on the section $efgh$,

We have:— $$f_n = \frac{P_n}{\frac{A}{\sin \theta}} = \frac{P}{A} \cdot \sin^2 \theta = f \cdot \sin^2 \theta.$$

Similarly, we get:— $f_t = f \cdot \sin \theta \cdot \cos \theta.$

Strain.—When a piece of material, such as a bar of iron, is in tension or compression under the action of an applied force P, the bar will, in consequence, be lengthened or shortened by an amount depending on the extent to which it is *loaded*. The *ratio* which this *change* of length bears to the original length of the bar is called the *longitudinal strain* due to P. Or in symbols,—

If,   L = Original length of bar in inches.

And, l = Change of length of bar also in inches.

We have :—   Strain $= \dfrac{l}{L}$ . . . . . . (II)

Since L and l are both actual lengths, measured by some common unit, the student should carefully note that strain, as thus defined, is merely an *abstract ratio*, and *not* a quantity, for it is independent of the units employed.

EXAMPLE I.—A tie-rod, 100 ft. long, is stretched $\frac{3}{4}$ of an inch by the action of a certain force ; what is the strain ?

Here,   L = 100 × 12 = 1,200 inches,

And, l = 0·75 inch.

∴   Strain $= \dfrac{0·75}{1,200} = 0·000625.$

Coefficient, or Modulus of Elasticity.—Experiment has demonstrated that for most materials used in engineering there is a very simple law connecting stress and strain, which is fairly well defined within certain limits. The stress is proportional to the strain, so long as the stress does not exceed a certain value, which, of course, is different for different materials and for different qualities of the same material. For example, if the stress be doubled, the strain will be doubled, or if the stress be reduced to one-half, the strain will also be halved, and so on. The limit beyond which this law does not hold is termed the **Limit of Elasticity.** When this limit is exceeded, the strain increases at a much greater rate than the stress producing it. Within the limit of elasticity, the material returns to its original state when the load is removed ; but when stressed

beyond this, the material does not do so, but retains a *permanent set.* In the following investigations the stress, in all cases, is assumed to be within the elastic limit :—

Consequently, $\dfrac{\text{Stress}}{\text{Strain}} = E$ (a constant). . . . . (III)

This E is termed Young's Modulus of Elasticity, or more appropriately by some writers the Coefficient of Elasticity.

Another way of exhibiting the relation subsisting among the various quantities we have been discussing, is to combine equations (I), (II), and (III) in such a way, as to express the stress and strain in terms of loads and dimensions.

Thus, $E = \dfrac{P}{A} \div \dfrac{l}{L}$;

Or, $PL = A\,l\,E.$ . . . . (IV)

Shear Strain and Stress.—Shear strain is the name given to the sliding movement of one layer of material over its neighbour, and the molecular force resisting this movement is called shear stress. If two plates be connected by a rivet and be pulled longitudinally, the rivet tends to be sheared through and is subjected to shear stress ; when a shaft is subjected to torsion (see p. 94), one section turns angularly slightly with respect to its neighbour, and therefore becomes subjected to shear stress.

Shear or Rigidity Modulus.—In the case of shear stress we have also for elastic materials a constant proportion of stress to strain ; the constant quantity is called the shear or rigidity modulus, and is defined in symbols by the relation—

Rigidity modulus = $C = \dfrac{\text{shear stress}}{\text{shear strain}}.$

According to one theory of elasticity, C is about $\frac{2}{5}$ E, and in practice it does not differ greatly from this.

Poisson's Ratio.—When a metal is stretched, it contracts laterally, and when compressed it expands laterally. The ratio $\dfrac{\text{transverse strain}}{\text{longitudinal strain}}$ is called *Poisson's ratio,* and is about $\frac{1}{4}$ for most materials.

Ultimate Strength.—The ultimate strength of a material is the intensity of stress which produces rupture. It is usually reckoned as so many lbs. or tons per square inch of the section of the material. It is always understood that the section taken is the original section of the bar, and not that at the instant and point of fracture.

Working Stress.—The working stress on a member of any structure is the maximum stress to which it is subjected in actual practice.

Factors ot Safety.—The ratio of the *ultimate strength* or limiting stress to the safe working stress is termed the *factor of safety* of the material. To determine the proper value of the "factor of safety," a number of considerations must be taken into account. A great deal depends on the quality of the materials and upon the manner in which different structures are stressed, such as the action and frequency of the working load.

1st. The load may be a constant *dead load*—*i.e.*, one which is steady and produces no appreciable vibration.

2nd. The load may be a *live load*, such as a regiment of walking soldiers, or a rolling load, in the case of a moving railway train, passing over a bridge.

3rd. Where the quality of materials is variable or liable to change, the factor of safety must be larger, than for more uniform materials and for those which are less affected by exposure to atmospheric and other deteriorating influences.

4th. In structures where the whole load cannot be ascertained with accuracy, the factor of safety must be increased, to allow for the unknown stressing actions, such as *impact due to passing trains on railway bridges.*

5th. In some structures, there is a liability to a sudden increase in the working load. Thus, in the case of a crane, where the weight may be allowed to descend rapidly and then be suddenly stopped, the maximum stress may be very much greater, than that due to the statical weight.

It is, therefore, necessary, that these special stresses be duly allowed for in the "factor of safety."

TABLE OF ULTIMATE STRENGTH AND WORKING STRESS OF MATE-RIALS WHEN IN TENSION, COMPRESSION, AND SHEARING.

Materials.	Ultimate Strength. Tons per sq. inch.			Working Stress. Tons per sq. inch.		
	Ten-sion.	Com-pression.	Shear-ing.	Ten-sion.	Com-pression.	Shear-ing.
Cast iron, . . .	7·5	45	14	1·5	9	3
Wrought-iron bars, .	20-25	20	20	5	3·5	4
Steel bars, . . .	45 *	70	30	9	9	5
Copper bolts, . .	15	25	...	3	5	...
Brass sheet, . .	14	...	...	3	...	...

* This is an average strength for steel rails with from 0·35 to 0·50 per cent. of carbon. With structural steel having from 0·18 to 0·25 per cent. of carbon, the usual specified value is from 28 to 32 tons per square inch.

TABLE OF FACTORS OF SAFETY IN COMMON USE.

Material.	FACTORS OF SAFETY FOR			
	A Dead Load.	A Live or Varying Load,* producing		In Structures subject to Varying Loads and Shocks.
		Stress of one kind only.	Equal Alternate Stresses of different kinds.	
Cast iron, and brittle metal and alloys, ...	4	6	10	15
Wrought iron and mild steel,	3	5	8	12
Cast steel, ...	3	5	8	15
Copper and other soft metals and alloys, ...	5	6	9	15
Timber, ...	7	10	15	20

To find the Factor of Safety for a Mixed Load.—"Given the proportions of *live* and *dead* load on a structure, to find the factor of safety for a mixed load ; multiply the factor of safety for a dead load, by a number proportional to the dead part of the load, and the factor of safety for a live load by the number proportional to the live part of the load ; add together the products, and divide by the sum of the multipliers."—(*Rankine.*)

EXAMPLE.— In an iron bridge, suppose

Dead load : live load :: 5 : 4.

Then, from the above table, we get $(3 \times 5) + (5 \times 4) = 35$, and $35 \div (5 + 4) = 4$, as the factor of safety for mixed loads.

Work done in Stretching a Bar.—Resilience.—If a load of *gradually* increasing amount be applied to a bar so as to stretch it, the amount of actual stretch, or elongation of the bar will, with the limitations already specified, be directly proportional to the load producing it. A diagram might, therefore, be drawn to represent graphically the work done in stretching the bar, as explained in Lecture II. of Volume I. The area of the diagram would represent the work done. The load will increase uniformly from 0 to P. The mean value of the force doing the work is, therefore, $\frac{1}{2}$ P, and the stretch or displacement is *l*. Hence, we have for the work done :—

$$W = \tfrac{1}{2} P l.$$

* See "Fatigue of Metals," Chap. VI., for effect of varying loads.

But from equations (I) and (III)—

$$P = f A, \qquad \text{and } l = \frac{f L}{E}.$$

Hence,

$$\left. \begin{array}{l} W = \dfrac{f^2}{E} \times \dfrac{A\,L}{2} \\[3mm] W = \dfrac{f^2}{E} \times \tfrac{1}{2} \text{ volume of the bar.} \end{array} \right\} \quad \ldots \text{ (V)}$$

Or,

The work done is therefore proportional to the volume of the bar, or to its weight.

When the bar is loaded to its elastic limit, or *proof stress*, as it is sometimes called, then the *work done* in stretching it is termed the Resilience of the bar, and the ratio $\dfrac{f^2}{E}$ is its Modulus or Coefficient of Resilience.

Example II.—What is the resilience of a material? If a wrought-iron tie bar, 5 feet long and 3 inches in diameter, has a limit of elasticity of 15 tons per square inch, and a modulus of elasticity of 30,000,000 lbs. per square inch, what is its resilience?

Answer.—$f = 15 \times 2240$ lbs., $E = 30,000,000$ lbs. per square inch, $A = \dfrac{1}{4} \times \dfrac{22}{7} \times 3^2$ square inches, and $L = 5$ feet.

$$\therefore \text{ Resilience} = \frac{(15 \times 2240)^2}{30,000,000} \times \frac{\frac{11}{14} \times 3^2 \times 5}{2} = 665 \text{ ft.-lbs.}$$

Sudden Pull, or Live Load.—We have just seen that a *constant* force of $\frac{1}{2}$P lbs. acting through a distance of l feet will do the same amount of work in stretching a bar as would a load gradually increasing from zero to P lbs.; therefore, the strain produced by a *sudden pull* of $\frac{1}{2}$P lbs. is the same as that due to P lbs. applied gradually. It follows, therefore, that if P be applied *suddenly*, but without initial velocity, the strain and, therefore, the stress will be doubled.

Therefore, *the stress in a bar caused by a suddenly applied or live load is twice that caused by a gradually applied or dead load of the same amount.*

Shrunk Rings.—In the construction of built-up guns, the process consists in shrinking on a series of concentric rings, each ring griping the next inner one with a certain pre-determined tension.

The reason for this arrangement will be better understood when we come to deal with the strength of thick cylinders. The principles set forth in the preceding sections enable us to calculate the dimensions of rings to give a certain grip.

Let D = The external diameter of an inner ring.

    „   **d =** The internal diameter of the next outer ring.

    „   **f =** The required tension.

When the outer ring is shrunk on, its diameter is then D. The inner fibres of this ring are then stretched by an amount $\pi\,(D-d)$; and by definition, we have :—

$$\text{Strain} = \frac{\pi\,(D-d)}{\pi\,d} = \frac{D-d}{d}.$$

If **E** denote the modulus of elasticity of the material of the ring, then :—

$$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{\dfrac{D-d}{d}}$$

Hence, $$d = D\left(\frac{E}{E+f}\right).$$

EXAMPLE III.—The external diameter of an inner ring is 20 inches. Work out the diameter which the outer ring must have in order to grip the inner one with an initial tension of 8 tons per sq. inch. Take the modulus of elasticity as 30,000,000.

ANSWER.—Here D = 20 inches, and f = 8 × 2240 = 17,920 lbs. per sq. inch.

$$\therefore \qquad d = 20 \times \frac{30,000,000}{30,017,920} = 19\cdot98 \text{ inches.}$$

Strength of Thin Cylinders.—By *thin* cylinders are meant cylindrical vessels whose thickness is small compared with their diameter. The resistance which such vessels offer to forces tending to burst them, both longitudinally and circumferentially, is easily deduced as follows :—Consider a cylindrical ring, whose breadth is b inches, thickness t inches, and internal diameter is d inches. Let p denote the intensity of the internal pressure, in lbs. per sq. inch, tending to burst the ring, and f the induced stress within the material of the ring, also in lbs. per sq. inch.

Then the magnitude of the total internal force tending to tear asunder the ring at the ends of a diameter is $p\,d\,b$ lbs. And the

resistance which the ring offers to this bursting force is $2\,t\,b\,f$ lbs.

These being equal, we have :—

$$2\,t\,b\,f = p\,d\,b \quad \therefore \quad f = \frac{p\,d}{2\,t}. \quad . \quad . \quad . \quad (VI)$$

This result shows that the stress, in a circumferential direction, is independent of the length of the cylinder.

Whatever be the form of the ends of the cylinder—whether they be flat or hemispherical—the total force tending to cause rupture circumferentially is $p\,\frac{\pi}{4}\,d^2$ lbs. ; resisting this force, we have a ring of material whose total sectional area is $\pi\,d\,t$ sq. inches.

Let f_1 be the longitudinal stress due to the longitudinal bursting force ; then the total resistance is $\pi\,d\,t\,f_1$ lbs.

And

$$\pi\,d\,t\,f_1 = p\,\frac{\pi}{4}\,d^2$$

Hence,

$$f_1 = \frac{p\,d}{4\,t}. \quad . \quad . \quad . \quad . \quad . \quad (VII)$$

From this we see that:—

$$f_1 = \tfrac{1}{2}\,f.$$

So that in a cylindrical boiler, which comes within the category of thin cylinders, the stress in a longitudinal direction is only one-half of the stress circumferentially.

Helical Seams.—If we made a boiler of rings, joined together circumferentially, then, so long as the strength of those joints was greater than one-half that of the solid plate, the boiler would still be as strong as one without joints, because the solid plate longitudinally would still be weaker than the circumferential joints. When, instead of solid rings, these are made up of pieces joined together longitudinally, it is obvious that the strength of the boiler is determined entirely by that of its longitudinal joints, unless the circumferential joints are less than half as strong.

As a compromise, it has been proposed to have, instead of circumferential and longitudinal joints, one continuous seam running spirally, called a helical joint.
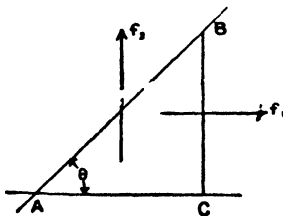
ILLUSTRATING STRESS ON HELICAL SEAMS.

Let the accompanying figure represent a portion of such a boiler **flattened cut.** A B is the helical seam, which, when flattened out, becomes a straight line, making the angle θ with the longitudinal direction. The longitudinal and circumferential stresses are represented by f_1 and f_2 respectively. The intensities of those stresses on A B being denoted by f_1' and f_2', we have :—

$$f_1' \times A\,B = f_1 \times B\,C; \qquad \text{and } f_2' \times A\,B = f_2 \times A\,C;$$

∴ $\qquad\qquad f_1' = f_1 \sin\theta; \qquad\qquad \text{and} \qquad\qquad f_2' = f_2 \cos\theta.$

Resolving f_1' and f_2' normally to A B, we have, for the total normal stress :—

$$f_n = f_1' \sin\theta + f_2' \cos\theta$$
$$\text{,, } = f_1 \sin^2\theta + f_2 \cos^2\theta.$$

But, $\qquad\qquad f_1 = \tfrac{1}{2} f_2.$

∴ $\qquad\qquad f_n = \tfrac{1}{2} f_2 \sin^2\theta + f_2 \cos^2\theta.$

Or, $\qquad\qquad \dfrac{f_n}{f_2} = 1 - \tfrac{1}{2}\sin^2\theta.$

Let, $\qquad\qquad n = \dfrac{B\,C}{A\,C};$

Then, $\qquad \sin^2\theta = \dfrac{B\,C^2}{A\,B^2} = \dfrac{B\,C^2}{B\,C^2 + A\,C^2} = \dfrac{n^2}{n^2 + 1}$

Hence, $\qquad \dfrac{f_n}{f_2} = 1 - \tfrac{1}{2}\cdot\dfrac{n^2}{n^2+1} = \dfrac{n^2+2}{2n^2+2} \quad \cdots \quad \text{(VIII)}$

For example, if $n = 1$, *i.e.*, $\theta = 45°$,

Then, $\qquad\qquad \dfrac{f_n}{f_2} = \dfrac{3}{4}.$

That is to say, that the normal stress on a spirally-running joint, making an angle of 45° with the axis of the boiler, would be three-fourths of that on a longitudinal joint. With joints of equal efficiency, therefore, the helical seam would be 33·3 per cent. stronger than the longitudinal one.

When the thickness of a cylindrical vessel, subjected to internal pressure is *not* small in comparison with its internal diameter, the problem requires to be treated differently. In this case, the stress will not be constant over the section, and cannot be investigated without more advanced treatment ; this we give later.

Chains.*—Chains are used both as transmitters of energy, like belts or ropes, and as simple fastenings. But, as these two modes of using chains are more or less connected, it is found convenient to treat them generally, without taking into account their special

---

* The following two figures, the Admiralty Rules for proof stress of chains, and the two Tables were obtained by permission from *Design of Structures*, by S Anglin, C.E., published by Charles Griffin & Co.

applications. Chains may be arranged under any one of the following headings :—1st, Round iron chains with open links; 2nd, round iron chains with stud links; 3rd, flat bar chains; 4th, gearing chains.

Chains are generally made from round wrought-iron bars of the best quality. The links are of different shapes, the elliptical form being the most common. They may be classed under two sorts—(a) The close or open link, which is commonly used for cranes and like purposes; (b) the stayed or stud-link, where a
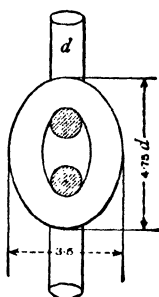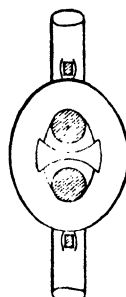
OPEN-LINK.   STUD-LINK.

stud or stay (usually made of cast iron) is fitted across the shorter diameter of each link. These studs most effectually keep the sides apart, and prevent any link jamming a neighbouring one. They add materially to the strength of the chain, for they are in compression whilst the sides of the links are in tension. Stud-links are used for large cables, and for other purposes where the chain may be subjected to heavy stresses.

When a load or tension is applied to any chain, each link is subjected to a bending action in addition to a tensile stress. This bending action is greatest at the extremities of the longer diameter of the link. Hence, on this reasoning, the link should be made stronger at its ends, but the question of the best section is complicated by the uncertainty as to the strength of the link at the *weld*. The diminution of strength due to "welds" in bars has been found to average 20 per cent. The strength of a stud-link may be taken as equal to double the strength of a rod of wrought iron, of the same diameter and quality of material as that of which the chain is composed; whereas, the strength of an open-link chain is only about 70 per cent. of this amount, even with perfect welding. It will easily be seen, that chains in passing over pulleys are subjected to other bending stresses than those enumerated above. Hence, the links of

chains to be used for this purpose should be made as *short* as possible, in order to increase the flexibility of the chain, and diminish this bending action.

Fatigue of Chains.*—Chains which have been in use for some time and subjected to many sudden jerks (such as the lifting chains for cranes and slings), suffer from "fatigue," the material becomes crystalline, or short in the grain, and consequently brittle and unsafe. The best precaution to adopt to restore the equilibrium of the material and re-establish its strength, is to periodically anneal the chains by drawing them very slowly through a fire, thus allowing them to become heated to a dull red, and then to cool them slowly in a heap of ashes. This method is carried out at Woolwich Arsenal by the War Department and by other high-class works.

Ultimate Stress, Proof Stress, and Working Stress on Chains.—There are three kinds of stresses to be distinguished when applied to chains—(1) The ultimate tensile stress, (2) the proof stress, and (3) the working stress. These are approximately related to each other as follows :—

The proof stress may be taken as one-half of the ultimate strength, and the working stress is often stated at about one-half the proof stress.

The Admiralty rules for the proof stress of studded-chain cables and close or open-link crane chains are as follows :—

Let d = diameter of the iron forming the chain ;

For studded chain cables—

Proof load in tons = $18 \, d^2 = 11\frac{1}{2}$ tons per sq. in. of section.

For close-link crane chains—

Proof load in tons = $12 \, d^2 = 7\cdot7$ tons per sq. in. of section.

If the **working load** be taken at one-half the proof load, we get—

For studded chains—

Greatest working load = $9 \, d^2 = 5\cdot75$ tons per sq. in. of section.

For close-link chains—

Greatest working load = $6 \, d^2 = 3\cdot85$ tons per sq. in. of section.

---

* The author was recently called upon to make a series of comparative tests between a $\frac{1}{2}$-inch sling chain, which had been in constant use during several years for lifting logs of wood without ever having been annealed, and a new well-annealed wrought-iron sling chain of the same size. The old chain broke with sudden snaps at 6·22 tons mean stress, and only 3·5 per cent. elongation, showing a brittle crystalline structure. The new well-annealed $\frac{1}{2}$-inch chain only gave way at 7·8 tons with 18·5 per cent elongation, showing a fine fibrous ductile structure. See also "Fatigue of Metals," Chap. VI.

Weight of Chains.—The weight of chains in lbs. per foot may be expressed from the equation, $w = 9\,d^2$, where d = diameter of iron in inches.

The following tables are given by Professor Unwin, M. Inst. C.E., in his *Elements of Machine Design*; also, in *Design of Structures*, by Mr. S. Anglin, C.E., Wh.Sc. The breaking strengths were calculated from the Woolwich experiments.

TABLE OF STRENGTH AND WEIGHT OF CLOSE-LINK CRANE CHAINS, AND SIZE OF EQUIVALENT HEMP CABLE.

Diameter of iron *d* in inches.	Weight of chain per fathom.	Breaking strength in tons.	Testing load in tons.	Girth of equivalent rope in inches.	Weight of rope in lbs. per fathom.
¼	3·5	1·9	0·75	2	1¼
₁₆⁵	3·0	3·0	1·10	2¼	1¾
⅜	8·5	4·3	1·6	3¼	2¼
₁₆⁷	11·0	5·9	2·3	4	3¾
½	14·0	7·7	3·0	4¾	5
₁₆⁹	18·0	9·7	3·8	5¼	7
⅝	24·0	12·0	4·6	6¼	8¼
₁₆¹¹	28·0	14·6	5·6	7	10¼
¾	31·5	17·3	6·8	7¼	12
₁₆¹³	37·0	20·4	7·9	8¼	15
⅞	44·0	23·1	9·1	9	17¼
₁₆¹⁵	50·0	26·1	10·5	9½	19½
1	56·0	29·3	12·0	10	22
1⅛	71·0	36·3	15·3	11¼	27¾
1¼	87·5	44·1	18·8	12¼	34¼
1⅜	105·8	52·8	22·6	13¾	41¼
1½	126·0	62·3	27·0	15	49¼

TABLE OF STRENGTH AND WEIGHT OF STUDDED-LINK CABLE

Diameter of Iron *d* in Inches.	Weight in Lbs. per Fathom.	Breaking Strength in Tons.	Test Load in Tons.	Girth of Equivalent Rope in Inches.	Weight of Rope in Lbs. per Fathom.
⅝	24	9·5	7	6¼	9
11/16	28	11·4	8½	7¼	12
¾	32	13·5	10¼	8	14
⅞	44	20·4	13¾	9¼	19¼
1	58	24·3	18	10½	22¼
1⅛	72	29·5	22¾	12	30¾
1¼	90	38·5	28¼	13¼	39¼
1⅜	110	48·5	34	15	48¼
1½	125	59·5	40¼	16	55
1⅝	145	66·5	47¼	17	62
1¾	170	74·1	55⅛	18	68¼
1⅞	195	92·9	63¼	20	86
2	230	99·5	72	22	104
2⅛	256	112	81¼	24	124
2¼	285	126	91⅛	26	145

For a thorough investigation of the strength of chain links, see a paper by Professors Goodenough and Moore, *University of Illinois Bulletin*, No. 18.

The Behaviour of Materials under Test.—*Stress-strain Diagrams.*—If a material be tested in tension or compression, and the strain at each stress be measured, and such strains be plotted on a diagram against the stresses, a diagram called the *stress strain diagram* is obtained. If a material obeys Hooke's Law, such diagram will be a straight line. For most metals, the stress-strain diagram will be a straight line until a certain point is reached, called the *elastic limit,* after which the strain increases

more quickly than the stress, until a point called the *yield point* is reached, when there is a sudden comparatively large increase in strain. After the yield point is reached, the metal becomes in a plastic state, and the strains go on increasing rapidly until fracture occurs.

Mild Steel.—The figure shows the stress-strain diagram for a tension specimen of mild steel, such as is suitable for structural work
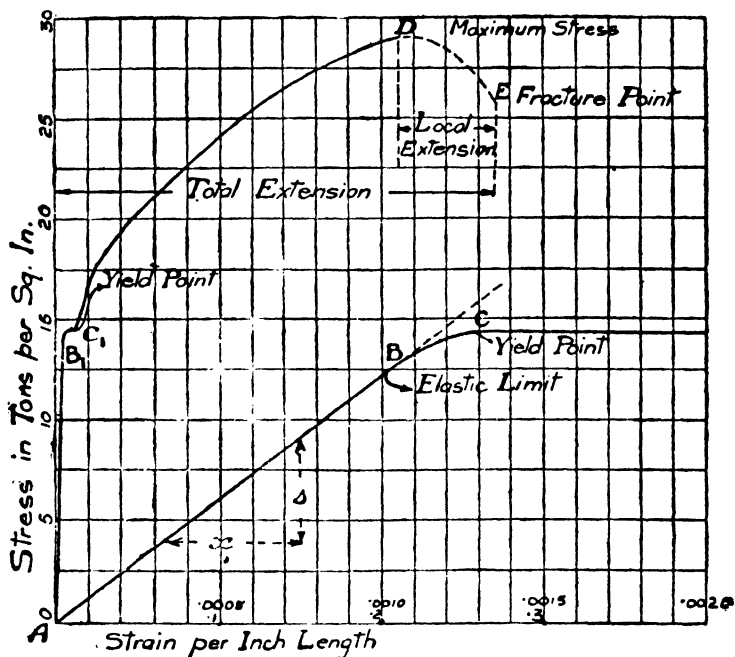
STRESS-STRAIN DIAGRAM FOR MILD STEEL.

The portion A B of the diagram is a straight line, and represents the period over which the material obeys Hooke's Law. At the point C, the yield point is reached, and the strain then increases to such an extent that the first portion

of the diagram is re-drawn to a considerably smaller scale, such portion being shown as A B_1C_1. The strain then increases in the form shown until the point D is reached, the curve between C_1 and D being approximately a parabola. When the point D is reached, the maximum stress has been reached, and the specimen begins to pull out and thin down at one section, and if the stress is sustained, fracture will then occur. The portion D E, shown dotted, represents increase of strain with apparent diminution of stress. This diminution is only apparent because the area of the specimen beyond the point rapidly gets smaller, so that the *load* may be decreased, and still keep the *stress* the same. In practice, it is very difficult to diminish the load so as to keep pace with the decrease in area, so that this last portion of the curve is very seldom accurate, and has, moreover, little practical importance in commercial testing because the maximum stress is always taken as that given at D.

The specimen draws down at the point of fracture in the manner shown in the diagram. Before the test, it is customary to make centre-punch marks at equal distances apart along the length of the specimen. The distance apart of these points after fracture of the specimen indicates the distribution of the elongation at different points along the length. Four such marks, *a*, *b*, *c*, *d*, are shown on the figure. The greatest extension occurs at the point of fracture, so that on a specimen of short length, the percentage total extension will be greater than on a longer specimen.

The stress-strain diagrams in compression and shear for mild steel are very similar to that for tension. In compression it is difficult to get the whole diagram, because failure occurs by *buckling*, except on very short lengths, where it is very difficult to measure the strains, and in shear the test is made by torsion, because it is almost impossible to eliminate the bending effect.

The importance of the elastic limit has been overlooked to a large extent by designers of machines and structures; but inasmuch as the theory, on which most of the formulæ for obtaining the strength of beams are based, assumes that the stress is proportional to the strain, it must be remembered that our calculations are true only so long as Hooke's Law is true, so that the elastic limit of the material is a very important quantity.

Tensile Strength of Various Steels and Wrought Iron.—The figure shows typical stress-strain diagrams for various kinds of steels and wrought iron; the strength properties depend to a large extent on the heat treatment and amount of "working" in manufacture.

Cast Iron.—The strength of cast iron varies considerably with its composition, but like all brittle material it is relatively weak in tension and strong in compression.

The figure shows the stress-strain diagrams for cast-iron in tension and compression, the results of the two tests being plotted in one diagram.
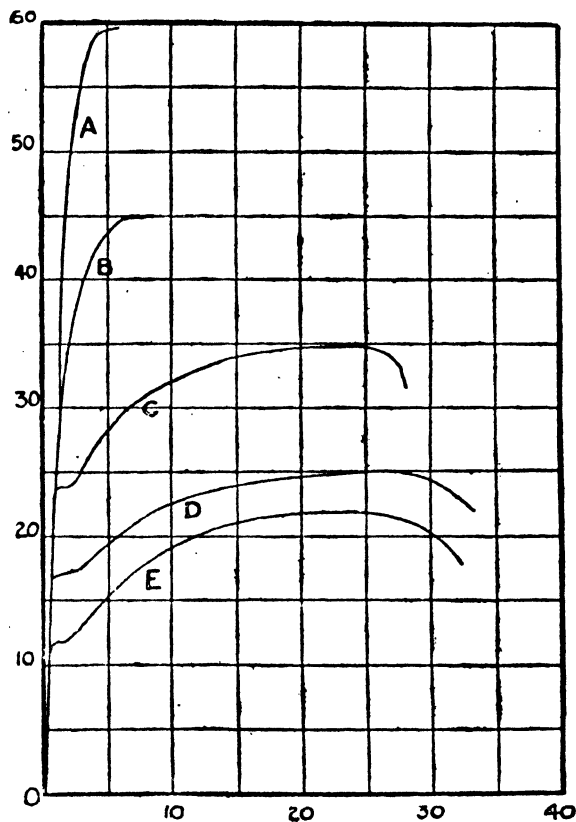
Extension per cent:—
A Tool Steel
(Unannealed).
B Crucible steel.

C Medium steel.
D Mild steel.
E Wrought iron.

STRESS-STRAIN DIAGRAMS FOR VARIOUS STEELS AND WROUGHT IRON.

It is clear from this diagram that the stress is never strictly

2

proportioned to the strain in tension; this has an important bearing upon the strength of cast-iron beams.

The compression diagram is not continued to failure as the failure would take place by buckling and injure the instrument for measuring the strain. When a cast-iron bar fails in tension, it breaks off "short"—*i.e.*, it does not produce a waist as indicated for mild steel.

When compression tests are made on cylinders which are so short that buckling effects are practically eliminated, the failure takes place by sliding diagonally and for shorter specimens still cracks sometimes develop which split off the outside portion leaving two inverted cones.
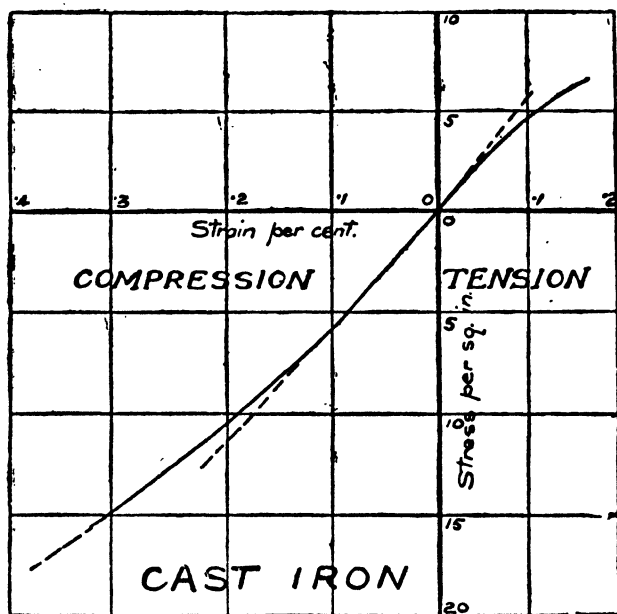
STRESS-STRAIN CURVES FOR CAST IRON.

The tensile strength of cast iron varies from about 7 to 15 tons per sq. in. in extreme cases, but more usually from 8 to 11 tons per sq. in. Figures for the compressive strength show more variation; this is probably due to the fact that the size of the test piece, both as regards its length and breadth, affects the result.

The following results of tests made upon ½-in. cubes by the

American Foundrymen's Association show that specimens cut from bars of small cross-section give much higher results than those from large.

Cross-Section of Bar from which Cubes were cut.	Crushing Strength in Tons per Square Inch for Cubes cut from				
	Middle Half-inch.	First Half-inch.	Second Half-inch.	Third Half-inch	Fourth Half-inch.
½ × ½	69·0	...	...	...	...
1 × 1	44·5	49·8	...	...	...
1½ × 1½	37·0	39·4	37·0	...	...
2 × 2	32·2	37·9	34·6	...	...
2½ × 2½	31·9	35·4	32·3	31·9	...
3 × 3	28·6	32·5	30·1	28·7	...
3½ × 3½	28·4	30·5	29·6	28·8	28·4
4 × 4	25·4	29·4	27·4	26·6	25·4

Stress-Strain Diagrams for various Ductile Metals.—The figure shows typical stress-strain diagrams for a number of
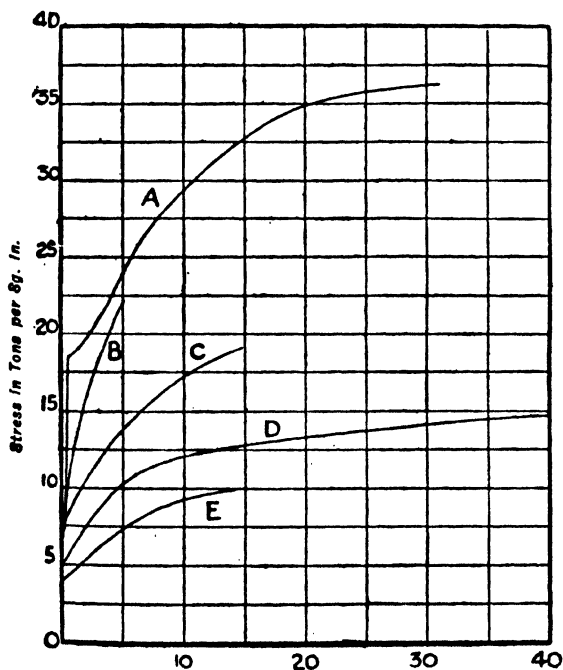
Extension per cent:

A Aluminium Bronze.

B Hard Brass.

C Annealed Brass.

D Rolled Annealed Copper.

E Rolled Aluminium.

STRESS-STRAIN DIAGRAMS FOR VARIOUS METALS.

ductile metals. These must be regarded as only average diagrams, because these materials vary in their elastic properties to a considerable extent, depending on the method of working and upon their constitution in the case of alloys.

In most cases the early portion of the stress-strain diagram is never quite straight, but there is usually a clearly defined yield point.

Stone, Concrete, Cement and like Brittle Materials.—When stone, concrete, cement and like materials are tested in compression in the form of cubes or short cylinders, fracture nearly always occurs by splitting in diagonal planes. This is commonly referred to as a "shear failure," the failure being attributed to the shear stresses on the diagonal planes at 45° to the axis.

Stress-Strain Diagram for Portland Cement and Concrete in Compression.—The kind of concrete which we will consider
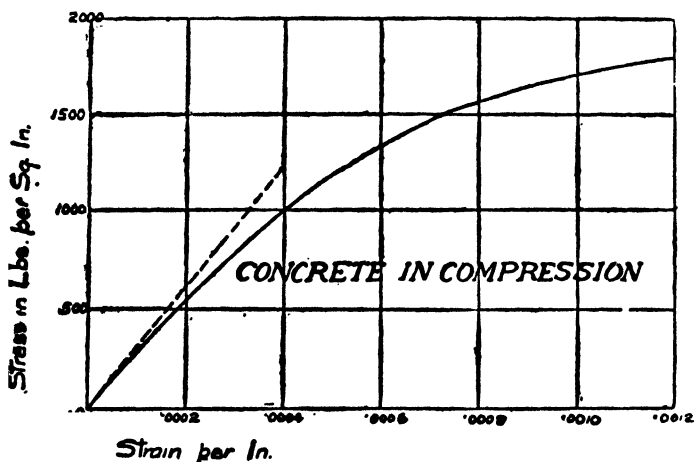
STRESS-STRAIN DIAGRAM FOR CONCRETE IN COMPRESSION.

is composed of a mixture of Portland cement, sand and broken stone or brick, gravel or like material which is called "aggregate."

The composition is usually referred to in the ratio of volumes of cement : sand : aggregate, i.e. a 1 : 2 : 4 ; concrete is one composed of 1 part cement, 2 parts sand and 4 parts aggregate.

The stress-strain diagram for concrete in compression is never quite straight so that there is no elastic limit, the exact

curve depending on the composition and on the time after setting.

The curve shown in the figure is almost exactly a parabola. This curve is for a 1 : 3 : 6 concrete, 90 days old, which was tested by Mr. R. H. Slocum, of the University of Illinois. Some authorities assume that the curve is a parabola, but in practice it is seldom that the curve comes so near to a parabola as the above. The stress-strain curve is, however, nearly always of a similar shape, the strains increasing more quickly than the stresses. It is extremely important to remember that with cement and concrete the relations between stress and strain vary largely with the quality and proportions of ingredients, and cannot be taken as almost constant as in the case of steel. In tension a somewhat similar curve is obtained, but as cement and concrete are practically never used in tension, much less work has been done on the tensile properties of concrete.

YOUNG'S MODULUS FOR CONCRETE.—In a material like concrete Young's modulus E is not constant, so that we must give the stress at which the ratio is taken if it is to have any real value.

The initial value of E is obtained by drawing a tangent to the curve at the origin as indicated in the figure.

We then have initial $E = \dfrac{1,200}{\cdot 0004} = 3 \times 10^6$ lbs. per square inch.

Similarly final $E = \dfrac{1,800}{\cdot 0012} = 1\cdot 5 \times 10^6$ lbs. per square inch.

The usual value of E taken in reinforced concrete calculations is 2×10^6 lbs. per square inch.

Effect of Composition and Age upon Compressive Strength of Concrete.—The compressive strength of concrete is roughly proportional to the proportion of the cement in the mortar, and increases with age, the increase being very rapid at first. One typical set of tests showed the following strength:—

Age, ...	7 days.	1 month.	3 months.	6 months.
Compressive Strength (lbs. per sq. inch), }	1,600	2,300	3,000	3,500

The following strengths are specified in the reinforced concrete regulations of the London County Council :—

Proportions by Volume.			Ultimate Compressive Strength in Lbs. per Square Inch.	
Cement.	Sand.	Coarse Material.	1 Month.	4 Months.
1	2	4	1,600	2,400
1·2	2	4	1,800	2,600
1·5	2	4	2,000	2,800
2	2	4	2,200	3,000

Tensile Strength of Portland Cement and Concrete.—The tensile strength of concrete is about $\frac{1}{10}$ of its compressive strength, but it is not usual to allow for any tension in the concrete in practice.

The standard method of testing the strength of neat cement is, however, by tension, so that the tensile strength is of considerable importance.

The British standard specification requires the following strength of Portland cement.

Briquettes 1 square inch in section must develop at least the following strength :—

Neat Cement.

After 7 days (1 in moist air, 6 in water)   -   - 450 lbs.
„ 28 „   „   „   27   „   -   - 500 „

$$\text{Breaking strength at 7 days} + \frac{40,000}{\text{Breaking strength at 7 days}}.$$

One part cement and three parts sand (by weight).

After 7 days (1 in moist air, 6 in water)   -   - 200 lbs.
„ 28 „   „   „   27   „   -   -   „

$$\text{Breaking strength at 7 days} + \frac{10,000}{\text{Breaking strength at 7 days}}.$$

LECTURE I.—QUESTIONS.

1. What do you understand by the terms *strain*, *stress*, and *modulus of elasticity?* A tie rod 100 feet long, and of 2 square inches sectional area, is stretched three-quarters of an inch under a tension of 32,000 lbs. What is the intensity of the stress, the strain, and the modulus of elasticity under these circumstances? *Ans.* 16,000 lbs. per square inch; 0·000625; 25,600,000.

2. A ship is moored by two cables of 90 feet and 100 feet in length respectively. The first cable stretches 2⅖ inches, and the second stretches 3 inches, under the pull of the ship; find the strain of each cable. *Ans.* ·00243; 0·0025.

3. Define the term *Resilience*. Show that the work done on a material by a *live load* is four times that done by an equal *dead load*. A wrought-iron tie rod 20 feet long and ·5 square inch cross-sectional area bears a dead load of 5,000 lbs. Find the work done on stretching the rod by this load. What live load would produce an instantaneous elongation of another $\frac{1}{10}$ inch? Take E = 30,000,000. *Ans.* 33·3 ft.-lbs.; 3,125 lbs.

4. A rod of iron 25 feet long and 2 square inches cross-sectional area checks a weight of 80 lbs., which falls from a height of 20 feet before beginning to strain it. Find the greatest stress and strain produced. Take E = 25,000,000. *Ans.* 40,000 lbs. per square inch; ·0016.

5. If the modulus of elasticity of a piece of steel in lbs. per square inch is 32,000,000, how much would a bar ⅔ of an inch in diameter and 25 inches long extend under a load of 10 tons? If its limit of elasticity is 21 tons per square inch, what is its resilience? *Ans.* 32·4 ft.-lbs.

6. What is the resilience of a bar? A bar of steel is ⅞ inch in diameter, and 30 inches in length, and is under a tensile pull of 10 tons, what is the work stored up in the bar, the modulus of elasticity being 32,000,000 lbs. per square inch? *Ans.* 391 inch-lbs.

7. Built-up guns are made of concentric rings, the outer hoops, or rings, being shrunk or forced upon inner tubes with a regulated tension. Supposing the external diameter of the inner tube to be 12 inches, and that the substance of its covering hoop is to have given to it an initial grip of 4 tons per square inch of its sectional area; the exterior diameter of this second hoop is 18 inches, and it is to be covered with a third hoop, having an initial grip of 8 tons per square inch of its sectional area; will you work out in arithmetic the difference of dimensions that will afford the above conditions? *Ans.* 11·99 internal diameter of covering hoop, 17·99 of third hoop.

8. Prove that when a thin spherical shell is exposed to the bursting pressure of gas or liquid the stress in the material is half as great as that within the curved surface of a thin cylindrical shell exposed to the like pressure, each shell being of the same thickness and diameter.

9. A long thin pipe of given internal radius is subjected to fluid pressure; find the tension of the material of the pipe. If the internal radius of the pipe is 6 inches, and the thickness of the pipe 0·5 inch, what fluid pressure per square inch would increase the radius of the pipe by 0·001 inch? The modulus of elasticity being 20,000,000, and the elasticity of the material being supposed to continue perfect. *Ans.* 277·7 lbs. per square inch.

10. A *steel* hydraulic cylinder, 10 feet long and 6 inches in diameter, acts as a brake on a lift. It has a movable piston fitted with a spring valve, the cylinder being full of liquid when the lift is at its highest position, and the piston and rod at the end of the stroke inside the cylinder. It was found that when the lift began to descend the internal pressure was 1,000 lbs. per square inch, which gradually rose to 2,000 lbs. when the piston had travelled 9 feet. Treating the cylinder as a thin one, what would be the law of variation of thickness at different points? Prove the formula.

11. A uniformly heavy chain is suspended from two given points: find the equation to the curve in which it hangs, and the tension at any point of the curve.

12 Prove that the tendency of a thin cylindric pipe to burst laterally (neglecting the strength of flanges, &c.) is twice as great as to burst endwise.

A wrought-iron pipe is 2 feet diameter, $\frac{1}{2}$ inch thick, its working stress is 5 tons to the square inch, but strength of plate is diminished 30 per cent. because of riveted joint. What is the working pressure? What head of water does this correspond to? *Ans.* 327 lbs. per square inch; 753 ft. head.

13. Prove the law for the tensile stress produced in a thick cylinder by internal fluid pressure. Describe how we attempt by chilling to give maximum strength.

14. A steel tube 5 inches internal and 7 inches external diameter has steel strip wound on it to the external diameter of 12 inches under a constant winding tensile stress of 15 tons per square inch. What is the stress at any place in the solid metal or the winding? *Ans.* 140 tons per square inch.

15. If a thin vessel is subjected to fluid pressure, p, inside (in excess of the outside pressure), prove that the total bursting force at any plane section is pA if A is the area of the whole section. How do we calculate the tensile stress in the section a of the metal? Find the rule for the stress in a thin spherical vessel. Does the rule apply to a thick vessel? Give reasons for your answer.

16. State clearly how we arrive at a rule for the proper thickness of a pipe, proving any formula used by you. In fixing the proper value of the stress that the material (say cast-iron) will stand, why do we not use the results of experiment on cast-iron test pieces in tension?

17. Knowing the axial and lateral strains in a tie bar of homogeneous material when subjected to a tensile force, show how we calculate the modulus of rigidity of the material.

18. A tube 3 inches internal and 8 inches external diameter is subjected to a collapsing pressure of 5 tons per square inch; show by curves the radial and circular stresses everywhere. Prove your formulæ. The limits of elasticity are not supposed to be exceeded.

19. A horizontal circular tube of steel is 7 feet diameter, $\frac{1}{2}$ inch thick, 100 feet long supported at the ends, its total load distributed uniformly all over being 30 tons, what are the greatest stresses in the metal? The tube is filled with compressed air, what must its pressure be if there is just no compressive stress in the metal? State what is now the nature of the stress in the metal at the place where it is greatest. *Ans.* 14,600 lbs. per square inch stress; 872 lbs. per square inch pressure; 29,200 lbs per square inch stress at bottom of tube.

LECTURE I.—C. & G. EXAM. QUESTIONS.

1. A steam engine has a piston 18 inches in diameter, and the greatest difference of steam pressure between the two sides of the piston when the engine is at work is 120 lbs. per square inch; what must the piston-rod diameter be in the body of the rod if the greatest intensity of stress per square inch is not to exceed 2,500 lbs? *Ans.* 4 inches diameter.

2. The links of a chain are made out of 1½-inch round bar iron, having a tenacity in pure tension of 22·5 tons per square inch: what load could be safely lifted with such a chain if the stress is not to exceed ½th of the breaking stress of the chain? You may assume that only three-quarters of the full tenacity would be developed in the material before rupture, when worked up into the form of a link. *Ans.* 8·5 tons.

3. How much would a steel tie-bar 3 inches in diameter and 25 feet 6 inches long extended under a total load of 33 tons? The modulus of elasticity of the steel is 12,500 tons per square inch. *Ans.* ·114 inch.

4. A cylinder 10 inches in diameter has a cover fixed on by 1½-inch studs. The internal fluid pressure is 200 lbs. per square inch above the atmospheric pressure. How many such studs would you employ if the tensile stress per square inch at the bottom of the threads is not to exceed 2,500 lbs.? (The diameter at the bottom of the thread of a bolt = 9·9 diameter of bolt − ·095 inch.) *Ans.* 9 studs.

1. A bar of steel 2 inches in diameter is subjected to a tension of 18 tons in a direction parallel to its axis. Find a general expression for the intensity of the normal and tangential stresses on a section inclined at any angle to the axis and calculate them for an angle of 30°. *Ans.* $f_n = f s m^2 \theta$; $f_t = \dfrac{r}{2} f s m^2 \theta$, where θ is the angle which the plane makes with the axis; $f_n = 1\cdot43$; $f_t = 2\cdot48$ tons per square inch.

2. Prove the formula for the stress on a longitudinal section of a thin cylindrical shell exposed to internal pressure. Can this formula be used for thick hollow cylinders? Give reasons for your answer.

3. What area and form of section would be suitable for a mild steel roof member subject to the following axial loads (+ signifying tension and − compression): (i.) due to dead load + 3,500 lbs.; (ii.) due to wind on one side + 6,000 lbs.; (iii.) due to wind on the other side − 4,000 lbs. *Ans.* 1 inch diameter.

4. Steel rails are welded together and there is no stress at a temperature of 60° F. They are prevented from buckling and they cannot expand or contract. Find the stresses when the temperature is (i.) 20° F., (ii.) 120° F. Steel expands by 0·0012 of its length for a change in temperature of 180° F. Take E = 30,000,000 lbs. per square inch. If the elastic limit in tension is 40,000 lbs. per square inch, at what temperature would it be reached? *Ans.* (i.) 8,000, (ii.) 12,000 lbs. per square inch; 260° F.

5. A piece of material is subjected to a pair of tensile stresses at right angles to each other of 8 and 4 tons per square inch respectively. Calculate the stresses produced on a plane making an angle of 30 degrees to the direction of the stress of 8 tons. *Ans.* 5 and 1·73 tons per square inch.

6. A 2-inch diameter bolt, 12 feet long, is screwed into the under side of a rigid beam, and hangs vertically. At the lower end of the bolt is a solid head; on the bolt is a weight of 100 lbs., which fits it loosely. The weight falls a distance of 10 feet on to the lower head. Neglecting the work done by the weight falling through the small distance due to the elastic stretch of the bar, calculate the stress produced in the bolt. Take the elastic modulus as 30,000,000 lbs. per square inch. *Ans.* 40,000 lbs. per square inch.

7. The internal diameter of a water main is 30 inches; it is constructed of steel plates ⅜ inch thick, with double-riveted butt joints. Assuming the tensile strength of the plates to be 30 tons per square inch, and the efficiency of the joints as 95 per cent., what is the bursting pressure of the main? *Ans.* 1,600 lbs. per square inch.

8. What is meant by the resilience of a body? Give an illustration.

9. Sketch the eye end of a steel link for a suspension bridge, the body of the link being 6 inches by 3 inches, and show how you determine the side of the pin. *Ans.* 6 inches.

10. Define the two terms "stress" and "strain." The suspension-rod of a bridge is 30 feet long and stretches $\frac{1}{20}$ inch under the load; what is the strain? *Ans.* Strain = ·000139.

11. Define the two terms "limit of elasticity" and "modulus of elasticity." A rod 100 feet long has a sectional area of 2 square inches, and, when submitted to a tension of 20 tons, stretches 0·62 inch; what is the modulus of elasticity of the material? *Ans.* E = 19,360 tons per square inch.

12. The limit of elasticity of a wrought-iron bar was found to be 20,000 lbs. per square inch, the strain at that point being 0·0006; what was the resilience of the material? *Ans.* 6 in.-lbs. per cubic inch.

13. Sketch the forged eye end of a steel tension bar, 4 inches by 1 inch, to some recognised proportion, and state generally what effect the diameter of the pin has upon these proportions. Assume the diameter of the pin to be three-fourths of the width of the bar.

14. A rod of steel 10 feet long with a sectional area of $\frac{1}{2}$ square inch, is kept at the proof strain with a tensile load of 25,000 lbs. ; what is the amount of the resilience of the rod when E = 35,000,000 lbs. per square inch? *Ans.* 23·65 feet.

15. A load of 3 cwts. is suspended by three parallel wires of equal length, each $\frac{1}{10}$ square inch in area, the outer ones being of steel and the inner one of brass, the wires being adjusted so that each has the same stress. A load of 3 cwts. is then added to the original load; what is the final stress on the wires, assuming E to be 30,000,000 lbs. per square inch for steel and 10,000,000 lbs. per square inch for brass? *Ans.* 1,600 lbs. per square inch in the brass; 2,560 lbs. per square inch in the steel.

16. A circular steel tank 20 feet diameter has to be filled with water to a depth of 14 feet. Assuming the efficiency of the joints to be 80 per cent., what must be the thickness of the plates so that a stress of 5 tons per square inch shall not be exceeded? Weight of water = 62·5 lbs. per cubic foot. *Ans.* ·08 inch.

17. State and explain fully the conditions which have to be taken into account in deciding on the thickness of the cylinder walls for a steam engine. Obtain a formula embodying these requirements, and compare the results it gives with the following empirical rule:—Thickness of wall = 0·03 $\sqrt{\text{diameter of cylinder}}$ × pressure, when the pressure is 200 lbs. per square inch gauge, and the diameter of the cylinder 30 inches, and taking 2,000 lbs. per square inch as the allowable stress in tension in cast iron.

18. A bar, 4 square inches in sectional area, carries a weight of 20 tons. Determine the intensities of the normal and tangential stresses on a plane making an angle of 60° with the cross-section.

19. Determine the area of cross-section required for a wrought-iron bar 16 feet long, in order that it may resist the energy of a load $\frac{3}{8}$ ton falling through a height of 6 inches, the resultant of the load being along the axis of the bar, and the material just strained to the elastic limit 12 tons per square inch. E = 12,000 tons per square inch.

20. A rod of steel 100 feet long, with a sectional area of 2 square inches, stretches $\frac{3}{4}$ of an inch when a load of 15 tons is suspended from the end ; determine the stress, the strain, and the modulus of elasticity.

21. A steel rod 30 feet long, having a sectional area of $\frac{1}{2}$ square inch, has a weight of one ton suspended from it ; what is the amount of energy stored in the bar owing to its extended condition, when the modulus of elasticity is 13,000 tons per square inch? If the weight were applied suddenly, what would be the amount of the energy stored ?

LECTURE II.

Strength of Beams and Girders.—The subject under this heading is one that naturally divides itself into two portions. (1) The determination of the resultant effects of the applied loads at any section of a beam or girder ; and (2) the nature and amount of the resistance offered by the beam or girder to rupture at that section.

When the section under consideration is in the same plane as the load, the only effect the load has at that section is a tendency to *shear* the beam ; but in the more general case, where the load acts at a distance from the given section, we have, in addition, a tendency to curve or bend the beam at the section. Hence the name *Bending Moment* is given to this latter effect.

In the accompanying figure, let A B represent a cantilever
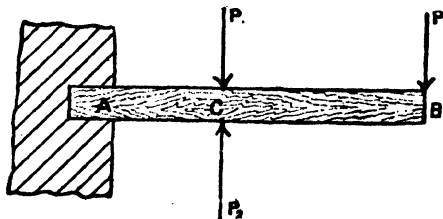
ILLUSTRATING SHEARING AND BENDING ACTION.

or beam fixed at one end, with a load P applied at the free end; and let C be any section in the beam. At C let there be applied two equal and opposite forces P_1, P_2, of the same magnitude as P. The introduction of these forces does not affect the equilibrium of the system, as P_1 and P_2 balance each other. Hence, the effect of P at the section C is equivalent to that of a

couple $P P_2$, with a single force P_1. A general proof of this important theorem is given in Vol. I., Lecture III , Prop. II. The couple constitutes the Bending Moment (B.M.), and the single force P_1, the Shearing Force (S.F.) at the section O.

DEFINITION.—The Shearing Force at any section of a beam is the algebraic sum of all the forces acting on either side of that section.

DEFINITION.—The Bending Moment at any section is the algebraic sum of the moments about that section of all the forces acting on either side of that section.

Or, in symbols, if P denote any one of the forces acting on one side of a section, and at a distance x, from it ; consider all the forces on the same side of the section as P, paying due regard to their sign—that is, if we reckon forces acting upwards as positive, we must regard those acting downwards as negative.

Then, $\text{S.F.} = \Sigma P.$
And, $\text{B.M.} = \Sigma Px.$ $\Big\}$ . . . . (I)

Beam Fixed at one end and Loaded at the other.—Let C D be a cross-section anywhere within the length of the beam at a distance of x inches from the fixed end A. To find the S.F. and B.M. at C D, we observe that the only force acting to the right of the section is W lbs. Hence :—

$$\text{S.F.} = W \text{ lbs.} \quad \ldots \ldots \ldots \text{(II)}$$
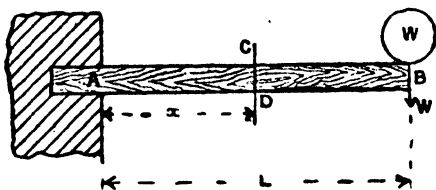
BEAM FIXED AT ONE END, LOADED AT OTHER.

It is independent of x, and therefore the same for all such sections as C D.

The B.M. at C D is W multiplied by its distance from the section in inches. Hence :

$$\text{B.M.} = W \times BD = W(L - x) \text{ inch-lbs.*} \quad . \quad . \text{ (III)}$$

This equation is true whatever may be the position of W on the beam, so long as L denotes its distance in inches from the fixed end, and C D is between W and the support.

* Where the older term "inch-lbs." has been used in connection with the Bending Moment at any section of a loaded beam, then the more modern term of "lb.-inches" may be substituted.

In this case, the diagram of the S.F. is a straight line parallel to the base and at a distance of W lbs. from it. Since (III) is the equation of a straight line, the B.M. is therefore a quantity increasing uniformly from zero, where x = L, to W L inch-lbs., where x = 0, as shown by the accompanying figure.

S.F. curve is a straight line.

B.M. curve is a straight line.
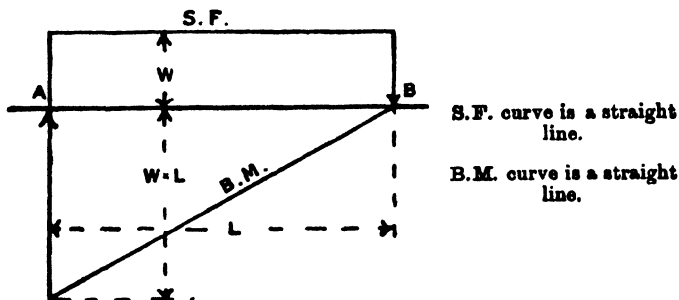
DIAGRAM OF S.F. AND B.M. FOR BEAM FIXED AT ONE END AND LOADED AT THE OTHER.

Beam Fixed at one end, and Loaded Uniformly.—Let the load on the beam be w lbs. per inch-run, it is required to find the shearing force and bending moment at any section C D, at x inches from the fixed end. As before, consider the part of the beam to the right of C D. The only force is the weight of that portion of the load carried by B D, so that :—

$$\text{S.F.} = w \times \text{B D} = w\,(\text{L} - x) \text{ lbs.} \quad . \quad . \quad . \quad \text{(IV)}$$
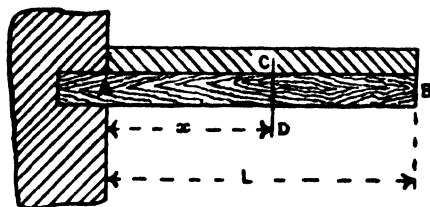
BEAM FIXED AT ONE END AND LOADED UNIFORMLY.

The moment of the portion of the load on B D with respect to C D is the same as if it were all concentrated at the middle point of B D. Hence :—

$$\text{B.M.} = w \times \text{B D} \times \tfrac{1}{2}\,\text{B D} = \tfrac{1}{2}\,w \times \text{B D}^2 = \tfrac{1}{2}\,w\,(\text{L} - x)^2 \text{ inch-lbs.} \quad \text{(V)}$$

Equations (IV) and (V) show us that both the S.F. and B.M. vanish when x = L; and when x = 0, we get :—

$$\text{S.F.} = w \text{ L lbs.} \quad \dots \quad \dots \quad (IV_a)$$

And,
$$\text{B.M.} = \tfrac{1}{2} w \text{ L}^2 \text{ inch-lbs} \quad \dots \quad \dots \quad (V_a)$$

The diagrams of S.F. and B.M. for this case take the forms shown in the accompanying figure.
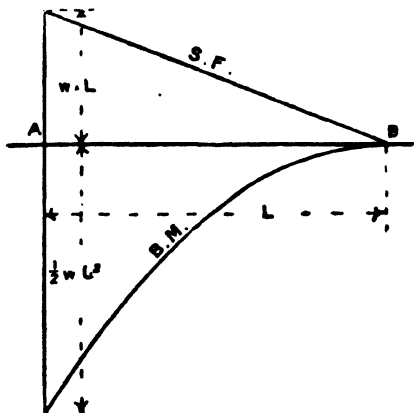
S. F. curve is a straight line.

B. M. curve is a parabola with vertex at B.

DIAGRAM OF S.F. AND B.M. FOR BEAM FIXED AT ONE END AND LOADED UNIFORMLY.

Beam Supported at both ends, and Loaded at the Middle.—In this case we measure x from the middle point of the beam. Since W is equidistant from A and B, the reactions at those points, R_1 and R_2, are equal to each other, and since their sum is W, we have :—

$$R_1 = R_2 = \tfrac{1}{2} \text{ W lbs.}$$
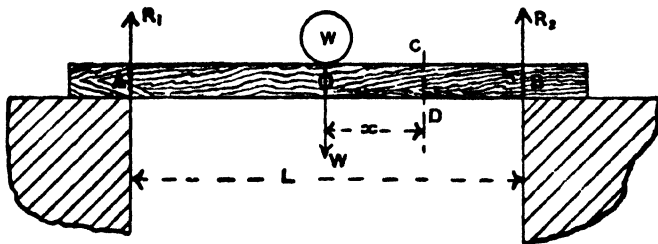
BEAM SUPPORTED AT BOTH ENDS AND LOADED AT MIDDLE.

The only force to the right of C D is R_2, and its leverage is B D.

Hence.   S.F. $= R_2 - \frac{1}{2}$ W lbs.   . . . . (VI)

And,   B.M. $= R_2 \times B D = \frac{1}{2} W (\frac{1}{2} L - x)$ inch-lbs.   . (VII)

Here, the B.M. vanishes when $x = \frac{1}{2}$ L, and increases uniformly from this until $x = 0$, when it attains its maximum value, $\frac{1}{4}$ W L inch-lbs.

Or,   Maximum B.M. $= \frac{1}{4}$ W L inch-lbs. . . . (VII$_a$)

The following figure shows the diagrams of S.F. and B.M. for this case :—
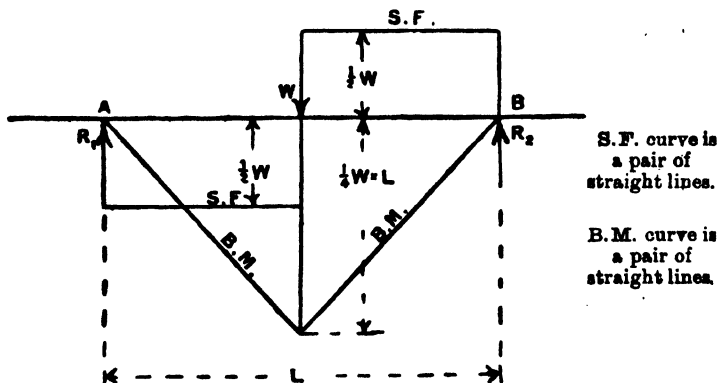
DIAGRAM OF S.F. AND B.M. FOR BEAM SUPPORTED AT BOTH ENDS AND LOADED IN MIDDLE.

EXAMPLE I.—In a beam of length L, supported at both ends and loaded at the middle with a load W, show that the bending moment is greatest at the centre of the beam and equal to $\frac{1}{4}$ W L. Then determine graphically the bending moment and shearing force at a point 6 ft. from one support in a beam of 25 ft. span loaded with 5 tons at its centre.

ANSWER.—We have already seen from equation (VII) that for a beam loaded as in this example, the B.M. at any distance x, from its middle point, is :—

$$\text{B.M.} = \frac{1}{2} W (\frac{1}{2} L - x).$$

This is obviously greatest when $x = 0$—that is, at the centre. Then :—

      Maximum B.M. $= \frac{1}{4}$ W L ; and S.F. $= \frac{1}{2}$ W.

EXAMPLE I.— LOADED BEAM. **33**

For the values of W and L given in the example, we get :—

Maximum B.M. $= \frac{1}{4} \times 5 \times 25 = 31 \cdot 25$ ft.-tons.

And, S.F. $= \frac{1}{2} \times 5 = 2 \cdot 5$ tons.
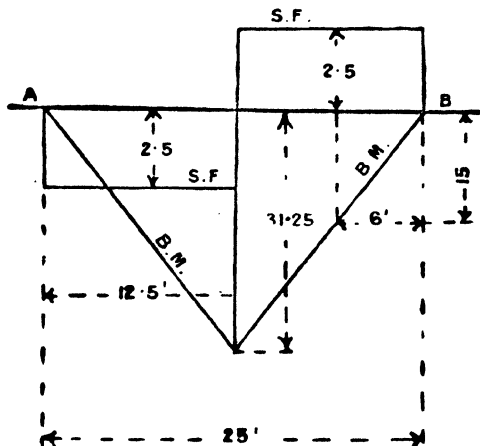
S.F. AND B.M. CURVES FOR EXAMPLE I.

The accompanying figure shows the diagrams of B.M. and S.F as constructed from these data.

At 6 feet from one end the B.M. measures 15 ft.-tons. This is easily verified by means of the formula for B.M., because $x = 12 \cdot 5 - 6 = 6 \cdot 5$.

∴ B.M. $= \frac{1}{2} \times 5 \times (12 \cdot 5 - 6 \cdot 5) = 15$ ft.-tons.

Beam Supported at both ends, and **Loaded Anywhere.**—With a single concentrated load, the maximum bending moment will
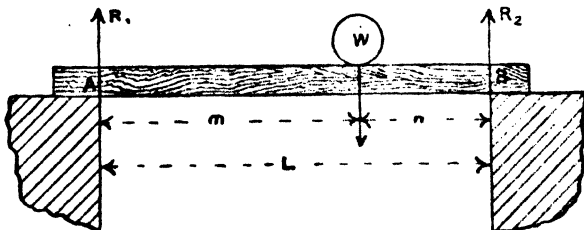
BEAM SUPPORTED AT BOTH ENDS, AND LOADED ANYWHERE.

always occur immediately under the load, whether it be at the middle of the beam or not.

For the B.M. at any section at a distance x, from one end is $R \times x$, and this is greatest when x is largest; that is, when the section is under the load.

To find the reactions at the supports, we take moments about A and B, and get $R_2 \times L = W \times m$.

$\therefore R_2 = \dfrac{m}{L} W$ lbs. and $R_1 = \dfrac{n}{L} W$ lbs. These are the values of the S.F. to the right and left of W respectively.

$$\left. \begin{array}{l} \text{S.F. (to right)} = \dfrac{m}{L} \text{ W lbs.} \\[2mm] \text{S.F. (to left)} = \dfrac{n}{L} \text{ W lbs.} \end{array} \right\} \qquad \text{(VIII)}$$

Multiplying the first of these equations by n, or the latter by m, we get :—

$$\text{Maximum B.M.} = \left(\frac{m\,n}{L} W \right) \text{ inch-lbs.} \quad . \quad . \quad \text{(IX)}$$

We can now construct the diagrams of S.F. and B.M. :—

S.F. curve is two straight lines.

B.M. curve is two straight lines.
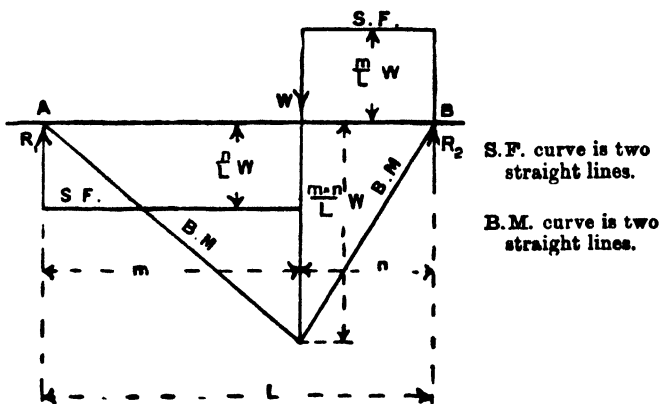
DIAGRAM OF S.F. AND B.M. FOR SINGLE LOAD IN ANY POSITION.

Beam Supported at both ends and Loaded Uniformly.—As before, let the weight per inch-run be denoted by w, then the total load carried by the beam will be w L lbs., and the reactions

R_1 and R_2, will each be $\frac{1}{2}\,w$ L lbs. Taking the forces to the right of the section C D.
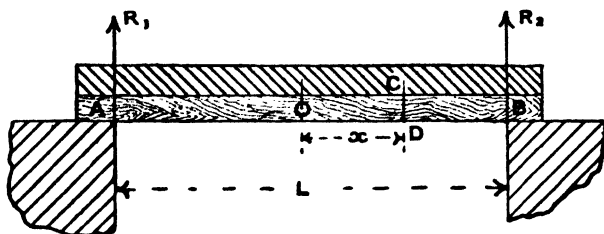
BEAM SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

We get, S.F. $= R_2 - w \times BD = \frac{1}{2}\,w\,L - w\,(\frac{1}{2}L - x) = w\,x$ lbs. (X)

And, B.M. $= R_2 \times BD - w\,.\,BD \times \frac{1}{2}\,BD$

,, $= \frac{1}{2}\,w\,L \times BD - \frac{1}{2}\,w\,.\,BD^2$

,, $= \frac{1}{2}\,w\,.\,BD\,(L - BD)$

,, $= \frac{1}{2}\,w\,(\frac{1}{2}\,L - x)\,(\frac{1}{2}\,L + x).$

$\therefore$ B.M. $= \frac{1}{2}\,w\,(\frac{1}{4}\,L^2 - x^2)$ inch-lbs. . . . . . (XI)

The limiting values of S.F. and B.M. are :—

When, $x = \frac{1}{2}\,L$; then, S.F. $= \frac{1}{2}\,w\,L$ lbs.; and, B.M. $= 0$. (X_a)

When, $x = 0$; then, S.F. $= 0$; and :—

Maximum, B.M. $= \frac{1}{8}\,w\,L^2$ inch-lbs. . . (XI_a)

Plotting our diagrams of S.F. and B.M., we get the figure shown on next page.

When a beam carries more than one load, or is loaded in more ways than one, the simplest and safest way is to consider each load separately, without regard to the others, and then combine the separate effects so as to obtain the resultant action, as in Example II.

EXAMPLE II.—Draw the bending moment and shearing force diagrams for a beam 12 feet long, supported at both ends, and loaded with weights of 4 and 6 tons at distances of 3 and 8 feet respectively, from one end of the beam. Explain fully the mode of arriving at these diagrams.

ANSWER.—Measuring distances from the left end of the beam, and considering each load separately, we have, for the 4 tons, to the left of the load :—

$$\text{S.F.}_1 = \frac{n}{L} \text{ W} = \frac{9}{12} \times 4 = 3 \text{ tons.}$$

And, to the right of it :—

$$\text{S.F.}_1 = \frac{m}{L} \text{ W} = \frac{3}{12} \times 4 = 1 \text{ ton}$$
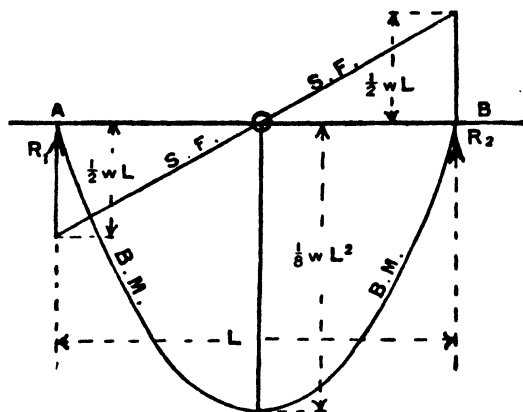
S.F. curve is a straight line.

B.M. curve is a parabola with vertex below the middle of the beam.

DIAGRAM OF S.F. AND B.M. FOR A BEAM SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

The maximum B.M.$_1$ due to this load is :—

$$\text{B.M.}_1 = \frac{m \times n}{L} \text{ W} = \frac{3 \times 9}{12} \times 4 = 9 \text{ ft.-tons.}$$

It occurs immediately under the load.

Next taking the 6-tons load, we have to the left of it :—

$$\text{S.F.}_2 = \frac{n}{L} \text{ W} = \frac{4}{12} \times 6 = 2 \text{ tons ;}$$

And to the right of it :—

$$\text{S.F.}_2 = \frac{m}{L} \text{ W} = \frac{8}{12} \times 6 = 4 \text{ tons.}$$

The maximum B.M.$_2$ due to the 6 tons is :—

$$\text{B.M.}_2 = \frac{m \times n}{L} \text{ W} = \frac{8 \times 4}{12} \times 6 = 16 \text{ ft.-tons.}$$

EXAMPLE II.—LOADED BEAM. **37**

Plotting these results, we get the accompanying figure :—
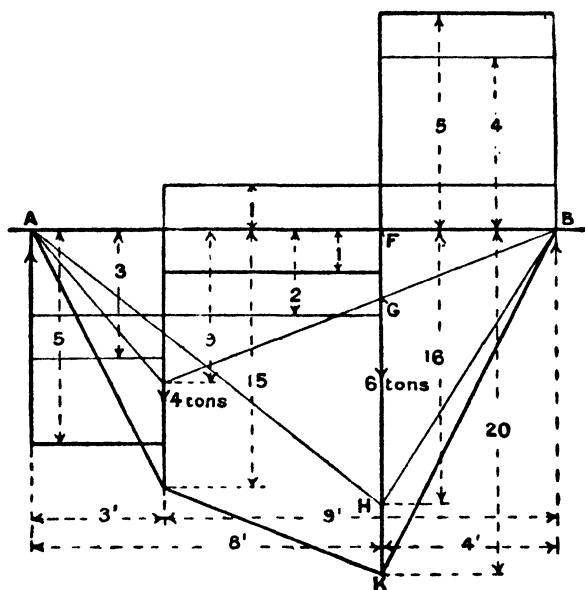
S.F. AND B.M. CURVES FOR EXAMPLE II.

The thin lines show the actions of the separate loads, and the full lines their combined results, obtained by taking the algebraic sum of the former.

The student should here carefully observe the necessity of attending to the *sign* of the shearing force. Thus, between the weights we have a shearing force of 2 tons, which, on account of its sign, is drawn below the base line ; also a shearing force of 1 ton drawn above the base line. The resultant shearing force between the loads is therefore the difference of these, and is drawn on the same side of the base line as the greater of its components.

The bending moments everywhere along the beam are of the same sign ; therefore, to obtain the combined bending moment diagram, we have simply to add the ordinates of each separate diagram. Thus, to get the total bending moment at the section under the 6 tons load, we add F G (viz., that due to the 4 tons at that point) to F H (that due to the 6 tons). The result F K is therefore the total B.M. at that point.

It is quite sufficient to do this for the sections under each load, and then to join each of the points so obtained with each other and with the ends of the beam by straight lines. If drawn to scale, the B.M. at any other point can then be obtained by measuring the corresponding ordinate.

EXAMPLE III.—A horizontal uniform bar, 18 inches long, is laid over two supports, each 4 inches from its ends. Find two points at which the bending moments are zero.

ANSWER.—Let w be the weight in lbs. per inch-run of the bar. Then the total weight of the bar will be $18\,w$ lbs., and the reactions will each be $9\,w$ lbs.
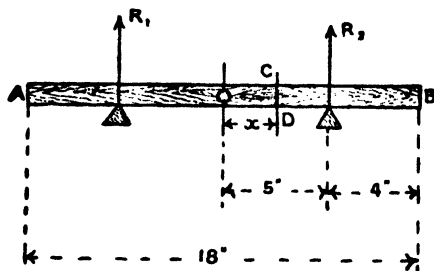
ILLUSTRATING EXAMPLE III.

Taking moments to the right of the section C D at a distance x inches from the centre of the bar, we get:—

$$\text{B.M.} = R_2(5 - x) - w \cdot BD \times \tfrac{1}{2} BD$$
$$\text{,,} = 9w(5 - x) - \tfrac{1}{2}w \cdot BD^2$$
$$\text{,,} = 9w(5 - x) - \tfrac{1}{2}w(9 - x)^2$$
$$\text{,,} = \tfrac{1}{2}w(9 - x^2) \text{ inch-lbs.}$$

The B.M. will be zero when $9 - x^2 = 0$; *i.e.*, when $x = \pm 3$ inches.

Hence, the required points are 3 inches on each side of the centre, or 2 inches inside of the supports.

Floating Beams.—When a *solid* body, such as a piece of wood of *uniform density*, floats in *still* water its weight and its buoyancy, or the resultant upward pressure of the water on the body, will at all points balance each other. There are consequently no shearing or bending stresses on the body, and each part is in equilibrium independently of the other parts.

But whenever those conditions are departed from, such as (1) when the floating body carries weights; (2) when it is not of uniform density, due to want of homogeneity in its material, if solid, or to its being hollow, or of a boat form; or (3) when it crosses waves, then bending and shearing stresses are set up.

Consider the case of a uniform beam of wood of rectangular section floating in still water. The beam will displace an amount of water exactly equal to its own weight. This is true, not only for the beam as a whole, but also for every individual segment of the beam. Any segment of the beam will displace just as much, and no more, water than it would do if floating by itself. The beam, therefore, is as free from stress as it would be if it were lying on a perfectly flat surface.

Suppose now that a weight W, be placed on the middle of a floating beam. This will cause the beam to sink to a greater depth and displace an *extra* volume of water. The *weight* of this *extra* displacement is exactly equal to W. What, now, is the condition of the beam as regards straining forces? Evidently, we need only consider the weight W, and the extra displacement, due to its being carried by the beam; because the upward reaction of the displacement due to the beam's own weight, is still at all points balanced by the downward weight of the beam. In other words, the condition of the beam, so far as its own weight and displacement are concerned, is in no way affected by the addition of the load.

To give definiteness to our ideas, let W be expressed in lbs., and let L denote the length of the beam in inches.

Then the forces we have to consider are :—

(1) W lbs. concentrated at the middle of the beam and acting downwards.

(2) The displacement of W lbs. of water uniformly distributed along the whole length of the beam and acting upwards, with an intensity of $\frac{W}{L}$ lbs. per inch of length.

The case is, therefore, analogous to that of a beam uniformly loaded and supported at its centre; or what is virtually the same thing, two beams of length equal to ½ L, fixed at one end and loaded uniformly. For, in order to obtain the shearing force and the bending moment at any section of the beam, x inches to either side of W, we have simply to substitute $\frac{W}{L}$ for w, and ½ L for L in equations (IV) and (V), and we get :—

$$\text{S.F.} = \frac{W}{L} (\tfrac{1}{2} L - x) \text{ lbs.} \quad . \quad . \quad . \quad . \quad \text{(XII)}$$

And, $$\text{B.M.} = \frac{W}{2\,L}\,(\tfrac{1}{2}\,L - x)^2 \text{ inch-lbs.} \qquad . \quad . \text{ (XIII)}$$

Under W the shearing force and bending moment are each a maximum. Their values may be found by making $x = 0$.

Then the   Maximum S.F. $= \tfrac{1}{2}$ W lbs. . . . (XII$_a$)

And the   Maximum B.M. $= \tfrac{1}{8}$ W L inch-lbs. (XIII$_a$)

The diagrams of S.F. and B.M. for this case are constructed in identically the same way as for a beam fixed at one end and carrying a uniform load, but taking $\tfrac{1}{2}$ L as a base line instead of L.

Suppose that, instead of one weight in the middle, the beam is loaded with two weights, one at each end, and each equal to W lbs., it is easy to see that the condition now is that of a beam uniformly loaded with $\dfrac{2\,W}{L}$ lbs. per inch-run and supported at each end. We have, therefore, only to apply formulæ (X) and (XI), substituting $\dfrac{2\,W}{L}$ for w, when we get :—

$$\text{S.F.} = \frac{2\,W}{L}\,.\,x \text{ lbs.} \qquad . \quad . \quad . \quad . \quad . \quad . \text{ (XIV)}$$

And, $$\text{B.M.} = \frac{W}{L}\,(\tfrac{1}{4}\,L^2 - x^2) \text{ inch-lbs.} \quad . \quad . \quad . \text{ (XV)}$$

Here the shearing force is a maximum when $x = \tfrac{1}{2}$ L, and the bending moment a maximum when $x = 0$.

Or,   Maximum S.F. $=$ W lbs. . . . . . (XIV$_a$)

And,   Maximum B.M. $= \tfrac{1}{4}$ W L inch-lbs. . . . (XV$_a$)

The diagrams of S.F. and B.M. are, therefore, in every way similar to those for a uniformly loaded beam supported at the ends.

Travelling Loads. —The simplest case of a movable load is that, wherein we are given a weight, say a heavy cylindrical body, rolling along a beam, to find the equations of maximum S.F. and B.M. for any position of the load, and exhibit these results in a diagram.

Referring to formulæ (VIII) and (IX), and the diagrams already deduced for a fixed load in *any* position on a beam, we

have for the maximum S.F. to the immediate right of the load :—

$$S.F. = \frac{m}{L} \ W \ \text{lbs.}$$

And, to the immediate left of the load :—

$$S.F. = - \frac{n}{L} \ W \ \text{lbs.}$$

For the maximum B.M., which occurs immediately under the load :—

$$B.M. = \frac{m \, n}{L} \ W \ \text{inch-lbs.}$$

Putting $m = x$ so that $n = L - x$, we obtain, when the load is x inches from the left end of the beam :—

The Maximum S.F. (*just to right of the section*) $= \dfrac{W}{L} x$

$\quad$ „ $\qquad$ „ $\qquad$ (*just to left of the section*) $= \dfrac{W}{L}(x-L)$ $\qquad$ (XVI)

And, $\quad$ Maximum B.M. $= \dfrac{W}{L} (L - x) x$ . . . . . (XVII)

To construct the diagram of S.F., we observe that its equation is that of a straight line, and that to the right of the section considered its value is zero when the load just starts from the left end of the beam, and increases uniformly as the load approaches the other end. That is :—

When, $\qquad x = 0$; then, S.F. $= 0$.

Also when, $\qquad x = L$; then, S.F. $= W$ lbs.

There is also another line for the shear at all positions just to the left of the load. This line passes through B, and its ordinate is $- W$ at the end A.

The equation of the B.M. curve is that of a parabola, whose axis is vertical, and passes through the middle point of the beam, where, of course, the maximum value of B.M. occurs. To construct this diagram, we have :—

When, $\qquad x = 0$, or $x = L$; then B.M. $= 0$.

Also when, $\quad x = \frac{1}{2} L$; then B.M. $= \frac{1}{4} W L$ inch-lbs.
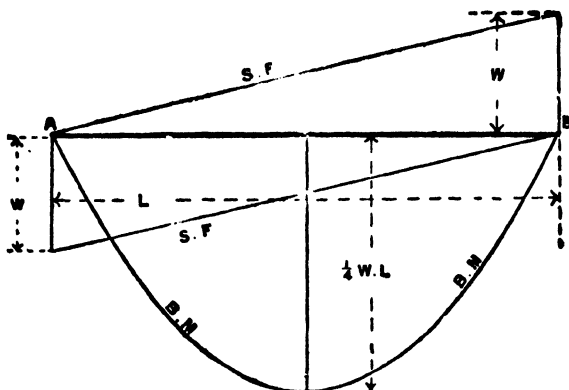
DIAGRAMS OF MAXIMUM S.F. AND B.M. FOR ROLLING LOAD.

Two Loads moving at a fixed distance apart.—From the above simple case we may easily pass to a very important practical example of moving loads—viz., overhead travelling cranes.

Here the crane rests on a carriage with four wheels running on two rails carried by girders, the weight of the whole machine together with the load being equally distributed over the wheels. Hence, considering one girder only, our problem is reduced to that of two equal loads moving along the girder at a fixed distance apart.
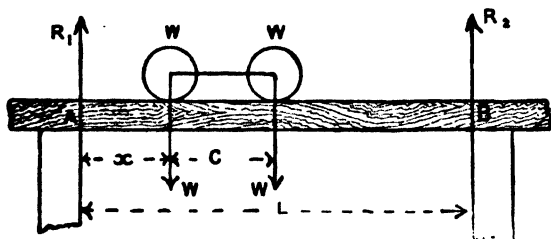
ILLUSTRATING TRAVELLING CRANE PROBLEM.

In the figure let W be the weight resting on each wheel, and c be the distance between their centres. If the motion be supposed to be in the direction shown by the arrow, it is evident that until the carriage gets to the middle of the girder the maximum

shearing force and bending moment will occur under the leading wheel—that is, if we estimate the shearing force to the immediate right of the wheel. But as the same thing takes place in the reverse order when the carriage moves from the other end of the girder in the opposite direction, we shall take the section of the girder immediately under the following wheel and estimate the shearing force and bending moment for that position. This method of procedure will be found to lead to simpler equations than if we had taken the leading wheel as our point of reference.

Now, considering the forces acting to the left of the wheel, we easily see :—

That, $\qquad$ S.F. $= R_1$.

And, $\qquad$ B.M. $= R_1 \times x$.

To find R_1 we take moments about B, which gives us :—

$$R_1 \times L = W\{L - (x + c)\} + W(L - x)$$

$$\text{,, \quad ,,} = W\{2(L - x) - c\}.$$

$$\therefore \qquad R_1 = \frac{W}{L}\{2(L - x) - c\}.$$

Hence, $\qquad$ S.F. $= \dfrac{W}{L}\{2(L - x) - c\}.$ . . . (XVIII)

And, $\qquad$ B.M. $= \dfrac{W}{L}\{2(L - x) - c\}x.$ . . . (XIX)

The equation for the S.F. is that of a straight line, and for the B.M. a parabola. To find the position and dimensions of those diagrams, we see that :—

When $\qquad x = 0$, S.F. $= \dfrac{W}{L}(2L - c)$, and B.M. $= 0$.

Again, both S.F. and B.M. will vanish when $2(L - x) - c = 0$; that is, when $x = L - \dfrac{c}{2}$.

To find the maximum ordinate of the B.M. curve, we have the condition that, when the B.M. is a maximum :—

$$\frac{d}{dx}(\text{B.M.}) = 0.$$

That is, $\qquad \dfrac{d}{dx}\{2(L - x) - c\}x = 0.$

Or, $\dfrac{d}{dx}\{(2\,L-c)x - 2x^2\} = 0.$

$(2\,L - c) - 4x = 0.$

Hence, $x = \dfrac{L}{2} - \dfrac{c}{4}\,.$

The shearing force diagram will, therefore, consist of two straight lines parallel to each other, and the bending moment diagram will consist of two equal parabolas intersecting at the middle of the girder. The axes of these equal parabolas will be equidistant from the middle of the girder and $\frac{1}{2}$ c units apart.

The following numerical example will elucidate this important case much better than a bare examination of formulæ :—

EXAMPLE IV.—In a travelling crane of 40 feet span the load is supported on a carriage which runs upon two similar girders, the axles of the carriage being 8 feet apart, and a load of $2\frac{1}{2}$ tons coming upon each wheel. Obtain a diagram showing the maximum bending moment at every section of the girder, and give the numerical values at distances of 10, 15, and 20 feet from one end. (Hons. S. & A. Exam., 1880.)

ANSWER.—Applying our general formulæ, we have, for the bending moment at any distance x ft., from one end :—

$$\textbf{B.M.} = \dfrac{W}{L}\Big[(2\,L-c) - 2x\Big]x.$$

Here, $W = 2\cdot5$ tons, $L = 40$ ft., and $c = 8$ ft.

∴ $\textbf{B.M.} = \dfrac{2\cdot5}{40}\Big[80 - 8 - 2x\Big]x.$

Or, $\textbf{B.M.} = \frac{1}{8}(36 - x)\,x$ ft.-tons.

For the numerical values asked for, we have :—

When $x = 10$ ft.; B.M. $= \frac{1}{8}(36 - 10)\,10 = 32\cdot5$ ft.-tons.

When $x = 15$ ft.; B.M. $= \frac{1}{8}(36 - 15)\,15 = 39\cdot375$ ft.-tons.

When $x = 20$ ft.; B.M. $= \frac{1}{8}(36 - 20)\,20 = 40$ ft.-tons.

We have seen, that the B.M. attains its maximum value when :—

$$x = \dfrac{L}{2} - \dfrac{c}{4} = \dfrac{40}{2} - \dfrac{8}{4} = 18 \text{ ft.}$$

Hence, the maximum B.M. $= \frac{1}{8}(36 - 18)\,18 = 40\cdot5$ ft.-tons.

The S.F. is not asked for in the question, but we here add it so as to make the example more complete.

The maximum S.F. occurs when $x = 0$, and when $x = L$, its value for this case then being :—

$$\text{Maximum S.F.} = \frac{2 \cdot 5}{40}(80 - 8) = 4 \cdot 5 \text{ tons.}$$

And, like the B.M., it is zero when $x = 40 - \dfrac{8}{2} = 36$ feet.

The following figure shows the S.F. and B.M. diagrams as required for this example.
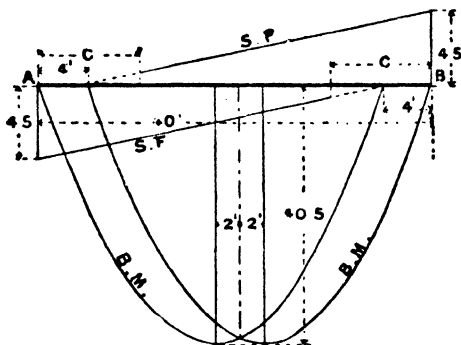
DIAGRAM OF MAXIMUM S.F. AND B.M. FOR A TRAVELLING CRANE.

Distributed Travelling Load.—The last case we shall consider is that in which a continuous load of uniform density, and long enough to completely cover it, comes on to a girder and moves off at the other end, such as a long train of uniform weight passing over a bridge.
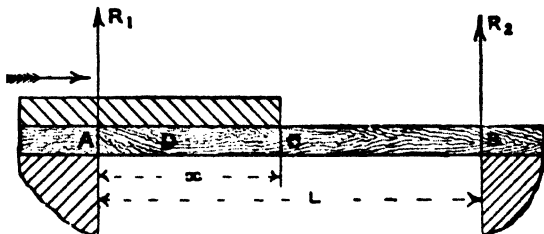
ILLUSTRATING TRAVELLING LOAD OF UNIFORM INTENSITY.

In the figure, let w denote the load per unit of length. When the load is in the position shown, it is clear that the S.F., at all points to the right of C, will be equal to R_2; and that at any section D, to the left of C, the S.F. will be less than R_2 by the weight of the portion of the load covering C D. It at once

follows that the S.F. is greatest at C, the front of the load, and this is true for all positions of C.

Hence, S.F. $= R_2$.

Taking moments about A, we have :—

$$R_2 \times L = w\,x \times \tfrac{1}{2}\,x.$$

∴ $$R_2 = \frac{w\,x^2}{2\,L}.$$

That is, S.F. $= \dfrac{w\,x^2}{2\,L}$ . . . . (XX)

The shearing force is, therefore, proportional to the square of the length of the part of the load resting on the girder.

The curve of the maximum bending moment is very easily deduced in this case. We have only to remember that the B.M. at *any fixed* section in the girder will get greater and greater for every additional part of the load that comes upon it; so that when the girder is wholly covered by the load the B.M. at every position will then be a maximum. The B.M. diagram is therefore identical with that given for a beam loaded uniformly, whilst the S.F. diagram becomes a parabola instead of a straight line.

The following figure shows how the S.F. and B.M. diagrams are constructed for this case.
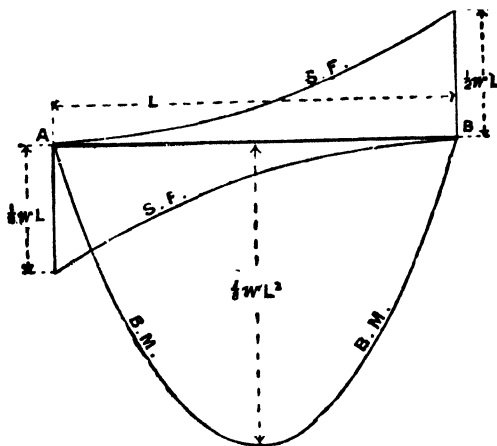
S.F. AND B.M. DIAGRAMS FOR TRAVELLING CONTINUOUS LOAD OF UNIFORM INTENSITY.

LECTURE II.—QUESTIONS.

1. Define "*bending moment*" and "*shearing force.*" A uniform beam weighing 15 cwts. rests on supports at its ends 20 feet apart. The beam is loaded with three weights of 4, 6, and 10 cwts. at distances of 2, 7, and 12 feet respectively from one of the supports. Find the B.M. and S.F. at a point 8 feet from the same support. *Ans.* B.M. = 98 ft.-cwts. ; S.F. = 3 cwts.

2. A bar of pine 48 inches long rests on props at its extremities, and just supports 7 weights, of 14 lbs. each, hung at equal intervals of 6 inches along the bar. Find the value of a single weight, which, if hung at the centre of the bar, would stress it to the same extent.

3. A batten of fir, 6 feet in length and supported at its extremities, will just sustain a load of 520 lbs. when hung at the centre. If this weight be removed, and two weights, each equal to P lbs., be hung at distances of 2 and 4 feet along the bar, what is the greatest value which may be assigned to P ? *Ans.* 390 lbs.

4. A beam, 20 feet long, whose weight is neglected, is supported at both ends and loaded with 1 ton evenly distributed along its length. Find the bending moment at a distance of 7 feet from one end. *Ans.* 5,096 ft.-lbs.

5. A beam, whose weight may be neglected, rests on supports at its ends 15 feet apart. Weights of 10, 6, 5, and 12 cwts. rest on the beam at intervals of 3 feet apart, the weight of 10 cwts. being 3 feet from one support. Find the points where the maximum bending moment and shearing force occur, and obtain their values. Construct the diagrams of bending moments and shearing force for the whole beam. *Ans.* The *max.* B.M. = 66 ft.-cwts., and occurs at all points between the weights 6 and 5 cwts.; the *max.* S.F. = 17 cwts., and occurs at the point where the weight of 12 cwts. rests.

6. A uniform cantilever, or beam fixed at one end and free at the other, 10 feet long, weighs 6 cwts., and carries two loads, one of 2 cwts. at the free end, and the other 4 cwts. at its middle point. Construct the shearing force diagram for the whole cantilever, and find the shearing forces at points 2½ feet and 6 feet from fixed end. *Ans.* 10·5 cwts.; 4·4 cwts.

7. A block of wood weighing 800 lbs., 20 feet long and 12 inches square, floats in water, and is loaded—
　　　　(1) By a weight of 200 lbs., placed at each extremity ;
　　　　(2) By a weight of 400 lbs. at the centre.
Show what forces act on the beam, and draw the curves of shearing force and bending moment for each case. *Ans.* (1) B.M. 1,000 ; S.F. 200 ; (2) B.M. 1000 ; S.F. 400 lbs. ft. units.

8. A girder is supported at both ends, and has a clear span of 30 feet. Show by means of a curve the position and magnitude of the greatest bending moment produced by a load of 20 tons as it rolls from one end to the other of the girder. Obtain the numerical results for distances respectively of 10 and 15 feet from one end. *Ans.* 133·3 and 150 ton-feet.

9. Prove an algebraic formula to show that, with a continuous load of uniform intensity passing over a beam A B such as when a long train passes over a bridge A to B, the maximum shearing stress to any point

K of the beam occurs when the part A K is fully loaded while the part K B is entirely unloaded, and that the magnitude of the stress is proportional to the square of the distance of K from the point A. A train of 1 ton per foot run, and upwards of 100 feet in length passes over a bridge of 100 feet span; what would be the maximum shearing force at distances of 25 to 50 feet respectively from one end of the bridge? Show how to determine graphically the shearing forces in the beam. *Ans.* Shear = 3·125 tons at 25 feet ; 12·5 tons at 50 feet.

10. Show how to obtain, and sketch the diagrams of maximum possible bending moment and maximum possible shear for a uniform rolling load of a given amount per foot run, as it passes over a girder of given span.

11. A girder of 22 feet span is supported at the two ends, a load of 10 tons rests on a point 2 feet from the left end, and two other loads of 6 and 7 tons respectively, at distances of 7 and 13 feet respectively from the first load : find the bending moment in inch-tons under each load, and also the shearing force. *Ans.* B M at C = 356$\frac{8}{11}$ ton-inches ; B M at D = 2,370$\frac{4}{11}$ ton-inches ; B M at E = 683$\frac{5}{11}$ ton-inches ; S F at C = 14$\frac{1}{11}$ tons ; S F between C and D = 4$\frac{1}{2}\frac{9}{2}$ tons ; S F between E and B = 8$\frac{4}{11}$ tons : S F between E and D = 1$\frac{3}{11}$ tons ; total at D = 6 tons.

12. A steel joint is used as a girder on a span of 17 feet 6 inches, and is freely supported at the two ends. It carries a uniformly distributed load of 18 cwts. per foot run, and two concentrated loads, one of 3 tons 3 feet from the left-hand support, and another of 1$\frac{3}{4}$ tons 10 feet from the same end. Find the bending moments in inch-pounds under each of the concentrated loads, and also at the centre of the girder, and also the shearing forces in pounds at each of these points. Sketch the bending moment diagram for the whole girder. *Ans.* B M under 3 tons = 780,444 lb.-inches ; B M under centre of girder = 1,122,660 lb.-inches ; B M under 7$\frac{3}{4}$ tons = 1,212,480 lb.-inches.

13. A girder, supported at the two ends, is 10 feet long, and is loaded uniformly with a load of $\frac{1}{2}$ ton per foot run, and also carries a weight of 3 tons placed 2 feet from one support. Sketch the curves at shearing force and bending moment, and find their numerical values at the centre of the span and at a section immediately under the concentrated load. *Ans.* B M at centre of span = 9·25 ton-feet ; B M under concentrated load = 8·8 ton-feet.

LECTURE II.—I.C.E. QUESTIONS.

1. **A log of wood, 20 feet long,** of uniform cross-section, floats in water **and carries two weights, each** equal to 1 cwt., at distances of 4 feet from **each end. Sketch, roughly to** scale, the diagrams of shearing force and bending moment on the log.

2. **A girder 20 feet long carries** a distributed load of 1 ton per lineal foot **over 6 feet of its length,** the load commencing at 3 feet from the left-hand **abutment. Sketch the shearing force** and bending-moment diagrams, and **find, independently, the magnitude** of the maximum bending moment and **the section at which it occurs.** *Ans.* 217 ton-feet, 13·6 feet from right-**hand support.**

3. **A girder, 40 feet long, rests** on two supports at distances of 10 feet **from the ends. It is loaded** with a uniformly distributed load of ½ ton **per foot run, and a concentrated** load of 10 tons at the centre. Draw, **to scale, diagrams of bending** moments and shearing forces.

4. **A cantilever 30 feet long** is built in at one end, and when unloaded **touches, but does not exert** pressure on, a support at the other end. It is **loaded with a uniformly distributed** load of 10 cwts. per foot run. Draw **diagrams of bending moments and** shearing forces, and find the position of **the point of contraflexure** (assume reaction at fixed end = ⅝ total load). *Ans.* B.M. = 56·25 ton-feet; shear = 9·37 tons at fixed end; shear at **other end = 5·62 tons;** point of contraflexure 7·5 feet from wall.

5. **A wire is carried across a** series of equal spans *a b*, *b c*, &c., of 240 feet **each, the weight of the wire** in each span being 400 lbs. The span *a b* has **a dip of 3 feet, and *b c* a** dip of 2·5 feet, the wire being attached at *b* to the **top of a vertical post 20 feet** high. Find the bending moment at the base **of the post.** *Ans.* 16,000 lb.-feet.

6. **A floating beam of timber** of uniform section and 20 feet long carries **three equal loads of 400 lbs.** each, one in the middle of its length, and one **at each end. Find the bending** moment at the centre, and sketch the **whole diagram of bending moments.** *Ans.* 1,000 lb.-feet.

7. **A beam 20 feet long is** supported on two supports, 3 feet from each **end of the beam. Weights** of 10 lbs. and 20 lbs. are suspended from the **two ends of the beam. Draw,** to scale, the bending-moment and shearing-**force diagrams; and, in particular,** estimate their values at the central **section of the beam.** *Ans.* B.M. at supports = 30 and 60 lb.-feet; at **centre = 45 lb.-feet.** Shearing forces 10, 2·14, 20 lbs.

8. **A girder crossing a span** of 408 feet is traversed by a railway train **having the uniform weight of** 1 ton per foot; and the train, whose length **is greater than the span,** may enter the bridge from either end. Find the **greatest positive and negative** values of the resulting shearing force at a **section 100 feet from either** abutment, and sketch the diagram of maxima. *Ans.* 112·5 tons; 12·5 tons.

9. **Show how the same question** can be determined when the train **consists of one or two heavy** locomotives followed by lighter carrying **stock.**

10. **Sketch approximately to scale** bending moment and shear diagrams **for a beam supported at its** ends and loaded half way along with a **uniformly distributed load. Show** how to find the maximum bending **moment.**

2   **4**

11. The side elevation of a cantilever 6 feet long is triangular, the depth increasing from 0 at the extremity to 12 inches at the wall into which it is fixed, the breadth is 3 inches, and the cross-section is rectangular. Sketch diagrams to show the distribution of tension, compression, and shear at the wall caused by its own weight.

12. A number of small web-plate girders were designed to carry a distributed test load of 100 tons each ; they were tested at the contractor's yard by placing two girders back to back about 3 feet apart, with slings at the ends, and applying a hydraulic jack between the girders at the centre, pressing against the two simultaneously with a pressure of 50 tons. Was this a fair test to all parts of the girder ? State the reasons for your answer. *Ans.* Yes, as regards strength.

13. A plank of wood, of specific gravity 0·7, 12 feet by 1 foot by 1 inch thick, rests in water in a horizontal position and carries concentrated loads of 5 lbs. at each end. What is the bending moment in the centre of the plank ? *Ans.* 180 lb.-inches.

14. A uniform iron beam, 10 feet long, weighing 600 lbs., rests on two rollers, the centres of which are 1 foot and 6 feet from one end. If the beam be rolled forward 3 feet, what will be the pressures on the two rollers in its new position ? *Ans.* 300 lbs. on each.

15. A beam 40 feet long resting on the ends is loaded with three weights of 5, 10, and 15 tons placed 10 feet apart, the 15-ton weight being at the centre of the span. Draw the diagrams of the bending moments and the shearing stresses. *Ans.* B.M. 162·5, 225, 137·5 ton-feet; shears 16¼, 6¼, 13¼ tons.

16. A beam, 20 feet span, supported at both ends, carries a uniform load of ½ ton per foot run, distributed over its length ; also two concentrated loads of 4 tons and 6 tons, at 4 feet and 12 feet respectively from the right support. Draw the curves of shearing force and bending moment for the whole span. Calculate the value of the bending moment at a section 9 feet from the right support.

17. A girder, 50 feet span, carries a uniform fixed load of ½ ton per foot run. A moving load, assumed of uniform weight, 1½ tons per foot run (of length greater than the span), travels over the girder from one end. Find the section at which the shearing force changes sign. Is this the section where the bending moment is greatest, and if not, where is it ?

18. A girder A B C D, 50 feet long rests on two piers, at B and C 30 feet apart, the ends A and C overhanging the piers by 10 feet. The 30 feet of girder between the piers B and C is loaded with a uniform load of 1 ton per foot run. At the end A is a weight of 12 tons, and at the end D a load of 8 tons. Draw the bending-moment and shearing-force diagrams for the whole girder. Calculate the bending moment at a section 12 feet from B in the portion B C.

19. A beam, 24 feet long, is uniformly loaded with 10 tons for three-quarters of its length, beginning at one abutment ; find the value and position of the maximum bending moment.

20. A road bridge of 60 feet effective span carries a travelling load of a vehicle on four axles, A, B, C, and D, A = 5 tons, B = 5 tons, C = 3 tons, and D = 3 tons. The distance between A and B = 7 feet 9 inches, between B and C = 7 feet 9 inches, and between C and D = 7 feet 9 inches, the total wheel-base being 23 feet 3 inches. Determine the position of the vehicle on the span when the bending moment under B is a maximum. State the amount of the bending moment.

21. Draw the bending-moment diagram for a continuous girder of four spans of 100 feet of uniform depth and section, and carrying a uniformly

distributed load of 1 ton per lineal foot on each end span, and 2 tons per lineal foot on each of the two spans on each side of the centre. State the amounts of the bending moments at each pier and at the centre of each span.

22. A sluice-gate consists of an outer frame 20 feet square, the top and bottom girders of which are connected by vertical ribs spaced 3 feet 4 inches apart, centre to centre, and covered by plating. The highest water-level coincides with the top of the gate. Calculate the load on one rib and determine the position and amount of the maximum bending moment and the amount of shear at each end. Fresh-water weighs 62½ lbs. per cubic foot.

23. A steel rail is 32 feet long and weighs 100 lbs. per yard. It rests on two supports, one at one end of the rail, the other at a point 10 feet from the other end of the rail. Find the position and amount of the maximum bending moment and shear when a weight of 200 lbs. hangs from the free end. Construct the bending-moment and shearing-force diagrams for this rail. *Ans.* Max. B M = 3,667 lb.-ft. ; max. shear = ± 533 at 10 ft. from one end.

24. A girder has a span of 40 feet, and two rolling loads of 10 tons and 15 tons respectively 10 feet apart pass over it. Find the maximum bending moment which can occur at any section and the maximum shear, and construct diagrams of maximum possible bending moment and shear. *Ans.* Max. B M = 302·5 ton-ft. ; max. shear 22·5 tons.

25. In a bridge, 120 feet span, with eight bays, the main girders are connected together by a trough flooring, on which a uniform live load of 2 tons per foot moves. Draw the positions of the moving load which would give a maximum stress in the braces.

26. Compare the loads which can be safely carried at the centre of a bar 12 feet long, 6 inches deep, and 2 inches broad, and of a girder of the same length and cubic contents whose overall depth is 14 inches and breadth of flange 4½ inches, thickness of web ¼ inch. *Ans.* Ratio = 1 : 5.

27. A girder 70 feet long carries a uniform load of 2 tons per lineal foot from one end to the middle, and a load of 20 tons at 20 feet from each end ; draw the bending-moment and shearing-force diagrams. *Ans.* Reactions 37·5, 72·5 tons.

LECTURE III.

Resistance of Beams to Flexure.—In the previous Lecture we saw that the effect of loading a beam was to give rise to both shearing and bending.

From the theory of couples set forth in Vol. I. we know that nothing but a couple can balance a couple. The resistance which a beam offers to bending must be of this nature, and therefore a couple of equal magnitude to that of the applied load, but of opposite tendency. The tendency of the applied couple is to bend or curve the beam, whilst the tendency of the induced couple is to oppose this curving action.

When a beam is curved the longitudinal fibres on the convex side of it are stretched beyond their normal length, and consequently they are in tension. On the concave side the fibres
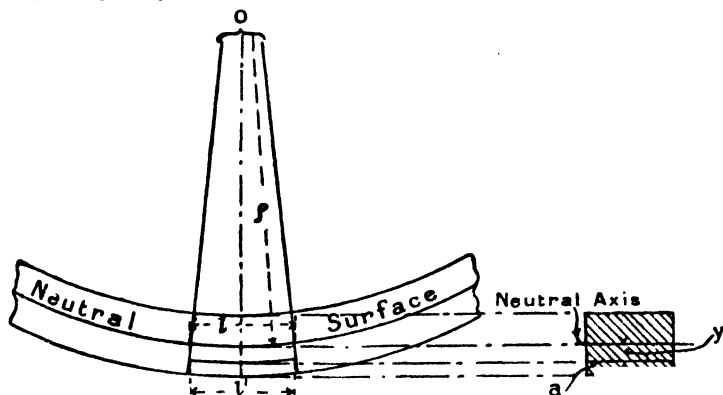
ILLUSTRATING FLEXURE OF BEAMS.

are shortened, and, therefore, they are in compression. Somewhere within the beam there must be a layer of fibres that are neither lengthened nor shortened, and are therefore unstressed.

This layer is termed the *neutral surface* of the beam, and the intersection of this surface with any cross-section of the beam is termed the *neutral axis* of that section. The neutral axis is of fundamental importance in the theory of beams, because it is the fulcrum about which both the bending and resisting couples act.

We shall now find the position of the neutral axis of any given section of a beam.

Let l be the length of a small portion of the neutral surface; l' that of a parallel layer of fibres on the stretched side of the beam, and at a distance y, from the neutral surface. If $l' = l$ when the beam is straight, it is evident that the amount of stretch in the fibres at distance, y, from the neutral surface will be $l' - l$, and the *strain* $\dfrac{l' - l}{l}$. Let ρ denote the radius of curvature of the neutral surface at the cross-section bisecting l. Then the radius of curvature corresponding to l' will be $= \rho + y$.

Hence,
$$\frac{\rho + y}{\rho} = \frac{l'}{l},$$

Or,
$$\frac{y}{\rho} = \frac{l' - l}{l}.$$

If f be the tensile stress at distance y, from the neutral axis, and E the modulus of elasticity of the material, we already know that :—

$$\frac{\text{stress}}{\text{strain}} = E,$$

Or,
$$\frac{f}{\frac{l' - l}{l}} = E.$$

Substituting $\dfrac{y}{\rho}$ for $\dfrac{l - l}{l}$, and inverting, we get :—

$$\frac{1}{\rho} = \frac{f}{E y}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot (I)$$

If we had considered in the same way a layer of fibres at a distance y', to the concave side of the neutral surface, and denoted the stress there as $-f'$ (the minus sign indicating compressive stress), we should have arrived at the equation :—

$$\frac{1}{\rho} = \frac{-f'}{E y'} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot (I_a)$$

Let a be a small element of the cross-sectional area at a distance y, then on the one side of the neutral axis we have for the total resistance to tension :—

$$\Sigma \, af = \frac{\mathrm{E}}{\rho} \, \Sigma \, a \, y. \quad . \quad . \quad . \quad . \quad \text{(II)}$$

On the other side of the neutral axis the total resistance to compression is :—

$$-\Sigma \, af' = \frac{\mathrm{E}}{\rho} \, \Sigma \, a \, y'. \quad . \quad . \quad . \quad . \quad \text{(II}_a\text{)}$$

But these forces constitute a couple, and are therefore equal. Hence, equating the right hand members, we have, neglecting the common factor, $\dfrac{\mathrm{E}}{\rho}$:—$\Sigma \, a \, y = \Sigma \, a \, y'.$

The neutral axis, therefore, passes through the centre of area of each cross-section. If, however, E be not the same for Tensile and Compressive stresses, then the N.A. will not pass through the centre of the area, but will lie to the side having the greater value of E.

¶To obtain the magnitude of the resisting couple, we multiply the resistances, af and af', by their respective distances, y and y', from the neutral axis, and sum up these products for the whole section. Thus, from equation (II) the total moment of resistance on the convex side of the neutral axis is :—

$$\Sigma \, afy = \frac{\mathrm{E}}{\rho} \Sigma \, a \, y^2;$$

And on the concave side :—

$$- \Sigma \, af' y' = \frac{\mathrm{E}}{\rho} \Sigma \, a \, y'^2.$$

The sum of these results constitutes the total Resisting Moment, R.M., for the section.

Hence,   R.M. $= \dfrac{\mathrm{E}}{\rho} \Sigma \, a \, y^2 + \dfrac{\mathrm{E}}{\rho} \Sigma \, a \, y'^2.$

There is now no longer any need for distinguishing between y and y', since the process of summation is the same all over the cross-section. We, therefore, finally get :—

$$\text{R.M.} = \frac{\mathrm{E}}{\rho} \Sigma \, a \, y^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

The quantity $\Sigma \, a \, y^2$, being a purely geometrical function, depending only on the *form* of the section, is termed its

Moment of Inertia, and is usually denoted by the symbol I, and sometimes by the product A k^2 (see Lecture XII., Vol. I.). Table II., Lecture XII., gives the values of k^2 for most of the sections required in the following examples. These multiplied by A will give the required values of I.

Writing I for $\Sigma a y^2$, our equations become :—

$$\text{B.M.} = \text{R.M.} = \frac{E\,I}{\rho} \,;$$

Or, the curvature, $\dfrac{1}{\rho} = \dfrac{M}{E\,I}.$ . . . . (IV)

Where M stands for either the B.M. or R.M.

Again, from equations (I) and (I_a), we get :—

$$\left.\begin{aligned} \frac{f}{y} &= \frac{E}{\rho} = \frac{M}{I} \\[2mm] M &= \frac{f}{y}\,I \\[2mm] f &= \frac{M}{I}\,y \end{aligned}\right\} \quad \ldots \ldots \text{(V)}$$

$\therefore$

Or,

Formulæ (IV) and (V) are the fundamental equations of the theory of the strength of beams and girders. In applying the latter equation, it must always be borne in mind that f stands for either the tensile or compressive stress at any distance y, above or below the neutral axis.

The greatest stress comes on the fibres farthest from the neutral axis, and is the principal effect to be considered in questions of strength. If this is amply provided for, the beam will be safe. Let y now denote the distance of the fibres farthest from the neutral axis :—

Then, $f_{max.} = \dfrac{M}{I} \times y = M \div \dfrac{I}{y}.$

The ratio $\dfrac{I}{y}$ is usually denoted by Z, and is called the **Modulus** of the Section.

Hence, writing Z for $\dfrac{1}{y}$, we have:—

$$\left. \begin{aligned} f_{max.} &= \frac{M}{Z} \\[2mm] M &= Z\,f_{max.} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(VI)}$$

Or,

In applying this equation the student must be careful to remember that in those cases where the section of the beam or girder is not symmetrical about the neutral axis, there will be two values of y to be taken into account, and therefore two values of Z. On the whole, we think it safer to adhere to the general formula (V) as being less likely to lead to confusion; at the same time, it is very convenient to use equation (VI) in taking out quantities in the drawing office by aid of tables since it reduces the work of calculation.

EXAMPLE I.—A floor joist, 12 inches deep and 3 inches broad, has a span of 15 feet, and carries a uniformly distributed load of 1 cwt. per foot-run. Find the greatest intensity of stress within the timber.

ANSWER.—In problems involving the calculation of stress within the beam, the student will find it best to express all dimensions in *inches*, and, therefore, bending moments in *inch-lbs.* or *inch-tons* as the case may be.

In this problem the greatest stresses will occur at the middle of the joist where the bending moment attains its maximum value, which, in this case, is:—

Max. B.M. $= \frac{1}{8}\, w\, L^2$ inch-lbs.

Here, $\qquad w = \dfrac{112}{12}$ lbs.

And, $\qquad L = 15 \times 12$ inches.

$\therefore \qquad$ B.M. $= \dfrac{1}{8} \times \left(\dfrac{112}{12}\right) \times (15 \times 12)^2$ inch-lbs.

Or, $\qquad$ B.M. $= 14 \times 15 \times 15 \times 12 \qquad$ „

The value of I for a rectangular section is:—

$$I = \frac{1}{12}(\text{breadth}) \times (\text{depth})^3,$$

Or, $\qquad I = \dfrac{1}{12} \times 3 \times 12^3 = 3 \times 12 \times 12.$

EXAMPLE II.—STRENGTH OF BEAMS. **57**

The greatest stress at the middle section of the joist will occur in the fibres farthest away from the neutral axis. Hence, $y = 6$ inches.

Applying equation (V) we have :—

$$f = \frac{\text{B.M.}}{I} y,$$

$$\text{,,} = \frac{14 \times 15 \times 15 \times 12}{3 \times 12 \times 12} \times 6 = 525 \text{ lbs. per sq. in.}$$

EXAMPLE II.—A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the centre. Find the greatest bending moment and the greatest stress in the fibres.

The specific gravity of oak is 0·934.

ANSWER.—Here the greatest bending moment takes place at the centre of the beam and is made up of two parts : (1) that due to the beam's own weight which is uniformly distributed along its length ; and (2) that due to the 5,000 lbs. concentrated at its middle.

For (1), B.M.$_1$ = $\frac{1}{8}$ w L^2 inch-lbs.

And for (2), B.M.$_2$ = $\frac{1}{4}$ W L ,,

∴ Total, B.M. = $\frac{1}{8}$ w L^2 + $\frac{1}{4}$ W L inch-lbs.

Taking the weight of a cubic inch of water as 0·036 lb., then a cubic inch of oak will weigh 0·934 × 0·036 = 0·0336 lb.

∴ $w = 0·0336 \times 15 \times 10 = 5·04$ lbs.

And, B.M. = $\frac{1}{8}$ × 5·04 × (10 × 12)^2 + $\frac{1}{4}$ × 5,000 × (10 × 12)

,, = 9,072 + 150,000 = 159,072 inch-lbs.

Here, I = $\frac{1}{12}$ × 10 × 15^3 = $\frac{1}{2}$ × 5 × 5 × 15 × 15

And, $y = \frac{1}{2}$ × 15 inches.

∴ $f = \dfrac{\text{B.M.}}{I} y = \dfrac{159,072}{\frac{1}{2} \times 5 \times 5 \times 15 \times 15} \times \frac{1}{2} \times 15$

,, = 424·1 lbs. per sq. inch.

EXAMPLE III.—A round steel spindle 10 inches long, and held at one end, revolves at the rate of 150 revolutions per minute round a vertical axis, to which the axis of the spindle is parallel

and from which it is 2 feet distant. The spindle has a uniformly
distributed load, the whole revolving weight being 30 lbs.
What should be the diameter of the spindle when the safe
working stress of the material in tension or compression is taken
at 25,000 lbs. per square inch?

ANSWER.—The spindle in this problem may be likened to a
beam fixed at one end and carrying a uniformly distributed load.
The load being not the revolving weight of 30 lbs., but the
centrifugal force of that weight due to its being whirled round
at the rate of 150 revolutions per minute.

$$\text{Velocity of spindle,} \; = \frac{150 \times 2\pi \times 2}{60} = 10\,\pi \text{ ft. per sec.}$$

$$\text{Centrifugal force,} \; = \frac{30 \times (10 \times \pi)^2}{32 \times 2} = \frac{1500 \times \pi^2}{32} \text{ lbs.}$$

This force, multiplied by half the length of the spindle, gives
us the bending moment at the fixed end of the spindle :—

$$\text{That is,} \qquad \text{B.M.} = \frac{1500 \times \pi^2}{32} \times \frac{10}{2} \text{ inch-lbs.}$$

If D be the diameter of the spindle in inches, then from Lecture
XII., Vol. I., we get :—

$$\text{The Moment of Inertia, I} = \frac{\pi}{64} D^4$$

$$\text{And the Modulus of Section, Z} = \frac{I}{\dfrac{D}{2}} = \frac{\pi}{32} D^3.$$

Now, $fZ = $ B.M. ; and, $f = 25,000$ lbs. per sq. inch.

$$\therefore \qquad 25,000 \times \frac{\pi}{32} D^3 = \frac{1500\,\pi^2}{32} \times \frac{10}{2}$$

$$\text{Or,} \qquad\qquad D^3 = 0.3 \times \pi = 0.94248$$

$$\text{Hence,} \qquad\qquad D = \sqrt[3]{0.94248} = 0.98 \text{ inch.}$$

EXAMPLE IV.—The section of a cast-iron girder, and the
maximum safe tensile and compressive stresses being given,
explain how to determine its moment of resistance to bending.
The dimensions of the section of a cast-iron girder are the

EXAMPLE IV.—FLANGED GIRDER. 59

following:—Top flange, 4 by $1\frac{1}{2}$ inches; bottom flange, 12 by $1\frac{3}{4}$ inches; web, 16 by $1\frac{1}{2}$ inches. Determine the moment of resistance, the greatest permissible tensile and compressive stresses being $2\frac{1}{2}$ and $7\frac{1}{2}$ tons per square inch respectively. If the girder be 20 feet long, and is supported at its two ends, find the greatest safe load which it will carry when uniformly distributed along its length.

ANSWER.—As this is an excellent example for showing the student how the R.M. of a girder section is calculated, we shall go into the matter in detail. Let the accompanying figure. represent the cross-section in question.

We have first to find the position of the neutral axis N A, by writing down the sectional areas of the parts composing the figure, and taking moments about the lower edge.
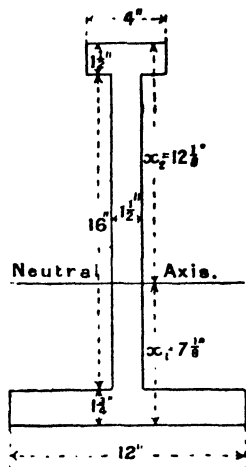
SECTION OF GIRDER.

Area of top flange $= 4 \times 1\frac{1}{2} = 6$ sq. in.

,, bottom ,, $= 12 \times 1\frac{3}{4} = 21$,,

,, web $= 16 \times 1\frac{1}{2} = 24$,,

∴ Total area of section $= 51$,,

Then, since N A passes through the centre of area of the section, we have :—

$$51 \times x_1 = 6 \times 18\frac{1}{2} + 24 \times 9\frac{3}{4} + 21 \times \frac{7}{8} = 363\frac{3}{8}.$$

∴ $x_1 = \dfrac{363\frac{3}{8}}{51} = 7\frac{1}{8}.$

And $x_2 = 19\frac{1}{4} - 7\frac{1}{8} = 12\frac{1}{8}.$

We calculate the value of I, the moment of inertia of the section about the neutral axis, by finding that for each of the parts into which the section is divided and taking their sum. As the neutral axis does not pass through the centre of any of

those parts, we shall have to employ Prop. I. of Lecture XII., Vol. I., to which we again refer the student.

Remembering that the moment of inertia of a rectangular area about an axis through its centre of gravity is :—

$$\tfrac{1}{12} \text{ (breadth)} \times \text{(depth)}^3, \text{ we have :—}$$

For top flange, I_t $= \tfrac{1}{12} \times 4 \times (1\tfrac{1}{2})^3 + 6 \times (11\tfrac{3}{8})^2.$

,,   ,,   ,, $= 1\cdot125 + 776\cdot343 = 777\cdot468.$

For bottom flange, $I_b = \tfrac{1}{12} \times 12 \times (1\tfrac{3}{4})^3 + 21 \times (6\tfrac{1}{4})^2.$

,,   ,,   ,, $= 5\cdot359 + 820\cdot312 = 825\cdot671.$

For web, I_w $= \tfrac{1}{12} \times 1\tfrac{1}{2} \times (16)^3 + 24 \times (2\tfrac{5}{8})^2.$

,,   ,, $= 512\cdot0 + 165\cdot375 = 677\cdot375.$

$\therefore$ For whole section, $I = 777\cdot468 + 825\cdot671 + 677\cdot375 = 2280\cdot5.$[*]

To illustrate what we said about the moduli of unsymmetrical sections, we shall find both moduli for this example :—

$$\text{For tension, } Z_t = \frac{I}{x_1} = \frac{2280\cdot5}{7\cdot125} = 320\cdot0.$$

$$\text{For compression, } Z_c = \frac{I}{x_2} = \frac{2280\cdot5}{12\cdot125} = 188\cdot0.$$

The question gives as the greatest permissible values for :—

Tensile stress, $f_t = 2\cdot5$ tons per sqr. inch.

Compressive stress, $f_c = 7\cdot5$,,   ,,

Since, R.M. $= Z f_{max.}$, we must take the lower of the two values of R.M. in fixing the load to be carried by the girder. These are :—

$$Z_t \times f_t = 320 \times 2\cdot5 = 800 \text{ inch-tons.}$$

And, $Z_c \times f_c = 188 \times 7\cdot5 = 1410 \text{ inch-tons.}$

$\therefore$ B.M. $=$ R.M. $= 800$ inch-tons.

---

[*] Another and rather shorter method of finding I for this form of section is to (1) produce the sides of top and bottom flanges to meet the neutral axis N A, (2) calculate the moments of inertia of the two full rectangles thus formed, (3) subtract from their sum the moments of inertia of the four rectangular areas which are in excess of the section of the beam. All these moments may be found by the formula $I = \tfrac{1}{12} B \times D^3$, which will only require to be used four times as the blank rectangles on each side of the web are equal in pairs.

The girder will, therefore. safely carry a uniformly distributed load, given by the equation :—

$$\tfrac{1}{8} \, w \, \mathrm{L}^2 = 800.$$

$$\therefore \qquad \mathrm{W} = w \, \mathrm{L} = \frac{8 \times 800}{20 \times 12} = 26\tfrac{2}{3} \text{ tons.}$$

This will make the maximum compressive stress

$$f_{c \; max.} = \frac{800}{188} = 4 \cdot 255 \text{ tons per sq. inch,}$$

instead of 7·5 as given ; showing that the girder is not well designed.

In a properly proportioned section we should have :—

$$Z_t \times f_t = Z_c \times f_c.$$

Steel I Beams.*— In the case of mild steel girders where the flanges are thin compared with their distance apart, and where the bending resistance of the web is practically negligible, formulæ for the moment of resistance are very simple.

Let A = Area of each flange.

$\quad$ D = Distance between centres of flanges.

$\quad$ f = Mean stress in each flange.

If the flanges are thin we can neglect the variation of stress over them and so may regard equal and opposite forces, each equal to f A, acting at the centre of the flanges.

These two equal and opposite forces form a couple whose moment (the moment of resistance) must be numerically equal to the bending moment since the beam is in equilibrium.

We thus have

$$f \mathrm{A} \, \mathrm{D} = \text{bending moment} = \mathbf{M} \; ;$$

i.e., $$f = \frac{\mathrm{M}}{\mathrm{A} \, \mathrm{D}}.$$

We could obtain a similar result from a consideration of the moment of inertia. By noting that, as the flange is thin and

* For properties of British Standard I Beams, *see* Appendix

the web negligible. we may write

$$J = 2A \times \left(\frac{D}{2}\right)^2$$

and

$$y = \frac{D}{2};$$

∴

$$f = \frac{My}{J} = \frac{M \cdot \frac{D}{2}}{2A\left(\frac{D}{2}\right)^2}$$

$$= \frac{M}{2A\left(\frac{D}{2}\right)} = \frac{M}{AD}.$$

EXAMPLE V.—A wrought-iron riveted girder of I section has a top flange of 9 square inches in sectional area, and a bottom flange of 8 square inches. The distance between the centres of gravity of the flanges is 12 inches, and the ends of the beam rest on abutments, 16 feet apart. The girder being loaded uniformly with a load equal to 1 ton per lineal foot (including the weight of the beam). What would be the mean stress per square inch on the metal in each flange at the dangerous section? The resistance of the web to bending is neglected.

ANSWER.—By "dangerous section" is here meant the middle section of the girder, where the maximum bending moment occurs. (See equation (XI_a) of Lecture II., Vol. II.)

Max. B.M. $= \frac{1}{8}\left(\frac{1}{12}\right) \times (16 \times 12)^2 = 32 \times 12$ inch-tons.

Hence, mean stress in tension flange,

$$f_t = \frac{32 \times 12}{8 \times 12} = 4 \text{ tons per square inch.}$$

And, mean stress in compression flange,

$$f_c = \frac{32 \times 12}{9 \times 12} = 3.55 \text{ tons per square inch.}$$

Curvature and Deflection of Beams.—When we speak of the curvature or the deflection of a beam we mean that of its neutral surface.

If the beam is fixed at one end, we take the origin of co-ordinates at that end; but if supported or fixed symmetrically at both ends, we take it at the middle.

Let the co-ordinates of the neutral surface curve be denoted, as usual, by x and y, then the deflection of the beam at any distance x, from the origin will be measured by y and the tangent of its inclination to the horizon by $\dfrac{d\,y}{d\,x}$.

The equation of the curve into which the beam is bent will be :—

$$y = \varphi\,(x),$$

Where $\varphi\,(x)$ is a function of x to be determined for each particular case.

In treatises on the analytical geometry of plane curves it is shown that the general expression for radius of curvature is :—

$$\rho = \frac{\left\{1 + \left(\dfrac{d\,y}{d\,x}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{d\,x^2}}.$$

Although of great importance, in the theory of beams $\dfrac{d\,y}{d\,x}$ is always such a small fraction, that its square becomes a perfectly negligible quantity in comparison with unity. We may, therefore, safely disregard the value of $\left(\dfrac{d\,y}{d\,x}\right)^2$ in the above formula, and write $\dfrac{d^2y}{dx^2} = \dfrac{1}{\rho}$.

But by equation (IV) of this Lecture we know that :—

$$\frac{1}{\rho} = \frac{M}{E\,I}. \quad \therefore \frac{d^2y}{dx^2} = \frac{M}{E\,I}. \quad . \quad . \quad . \quad . \text{(VIII)}$$

In what follows we shall assume that the beam or girder is of uniform section so that I is constant, the more general cases where I varies being rather beyond the scope of this treatise.

We shall begin by working out the following example, which will form a good introduction to this rather mathematical part of our subject.

EXAMPLE VI.—Investigate a formula for calculating the amount of deflection of a beam supported at its ends and loaded uniformly. Find the deflection in a beam of timber of uniform rectangular section, 6 inches wide and 12 inches deep, the beam

being supported at its ends in a horizontal position on two walls
12 feet apart.   There is to be taken into account a single con-
centrated load of 4,000 lbs. at the centre, and a uniformly dis-
tributed load of 2,500 lbs., the modulus of elasticity being
1,750,000 lbs. per square inch.

ANSWER.—Taking the middle of the beam as the origin of co-
ordinates, we have already proved (see equation (XI) in Lecture
II.) that the bending moment at x inches from this point, in
the case of a beam L inches between supports, and loaded
uniformly with w lbs. per inch-run, is :—

$$\text{B.M.} = \tfrac{1}{2} w (\tfrac{1}{4} L^2 - x^2) \text{ inch-lbs.}$$

Substituting this in formula (VIII) we get :—

$$\frac{d^2y}{dx^2} = \frac{w}{2 E I} (\tfrac{1}{4} L^2 - x^2).$$

Now, multiplying both sides by dx, and integrating, we
have :—

$$\frac{dy}{dx} = \frac{w}{2 E I} \int (\tfrac{1}{4} L^2 - x^2) \, dx.$$

Or,
$$\frac{dy}{dx} = \frac{w}{2 E I} (\tfrac{1}{4} L^2 x - \tfrac{1}{3} x^3).$$

This needs no correction because $\dfrac{dy}{dx} = 0$, when $x = 0$.

Integrating a second time, we get :—

$$y = \frac{w}{2 E I} \int (\tfrac{1}{4} L^2 x - \tfrac{1}{3} x^3) \, dx.$$

Or,
$$y = \frac{w}{8 E I} (\tfrac{1}{2} L^2 x^2 - \tfrac{1}{3} x^4). \quad . \quad . \quad . \quad . \quad \text{(IX)}$$

This also requires no correction, as x and y vanish together.
Now, let Δ_1 denote the deflection of the beam for the distri-
buted load :—

$$\Delta_1 = y, \text{ when } x = \tfrac{1}{2} L.$$

Or,
$$\Delta_1 = \frac{w}{8 E I} \{ \tfrac{1}{2} L^2 (\tfrac{1}{2} L)^2 - \tfrac{1}{3} (\tfrac{1}{2} L)^4 \}.$$

$$\therefore \qquad \Delta_1 = \frac{5 w L^4}{384 E I} \text{ inches.} \quad . \quad . \quad . \quad . \quad . \quad \text{(X)}$$

In the case of a beam carrying a single load of W lbs. at its

middle point, the bending moment due to that load at x inches from the middle point (Equation (VII), Lecture II.) is :—

$$\text{B.M.} = \tfrac{1}{2} W (\tfrac{1}{2} L - x) \text{ inch-lbs.}$$

$$\therefore \quad \frac{d^2 y}{d x^2} = \frac{W}{2 E I} (\tfrac{1}{2} L - x).$$

The first integration of this equation gives :—

$$\frac{d y}{d x} = \frac{W}{2 E I} (\tfrac{1}{2} L x - \tfrac{1}{2} x^2) = \frac{W}{4 E I} (L x - x^2).$$

And the second integration :—

$$y = \frac{W}{4 E I} (\tfrac{1}{2} L x^2 - \tfrac{1}{3} x^3). \quad \ldots \quad \text{(XI)}$$

If Δ_2 be the total deflection in this case, then Δ_2 is the value of y when $x = \tfrac{1}{2} L$.

$$\therefore \quad \Delta_2 = \frac{W}{4 E I} \{ \tfrac{1}{2} L (\tfrac{1}{2} L)^2 - \tfrac{1}{3} (\tfrac{1}{2} L)^3 \}.$$

Or, $$\Delta_2 = \frac{W L^3}{48 E I} \text{ inches.} \quad \ldots \ldots \quad \text{(XII)}$$

If the beam carry both loads at the same time, as given in the question, then the total deflection due to the two loads will be :—

$$\Delta = \Delta_1 + \Delta_2.$$

$$\therefore \quad \Delta = \frac{5 \, w \, L^4}{384 \, E I} + \frac{W L^3}{48 \, E I}.$$

Or, $$\Delta = \frac{L^3}{48 \, E I} (\tfrac{5}{8} w L + W).$$

The numerical data given are :—

$$L = 12 \times 12 \text{ inches.}$$

$$I = \tfrac{1}{12} b d^3 = \tfrac{1}{12} \times 6 \times 12^3 = 6 \times 12^2.$$

$$W = 4,000 \text{ lbs.}$$

$$w L = 2,500.$$

And, $$E = 1,750,000.$$

$$\therefore \quad \Delta = \frac{(12 \times 12)^3}{48 \times 1,750,000 \times 6 \times 12^2} (\tfrac{5}{8} \times 2,500 + 4,000)$$

$$\Delta = 0{\cdot}2288 \text{ inches.}$$

Uniform Beam on Three Supports.—Suppose we are given a uniform beam resting on three supports all on the same level, to find the pressure on the middle support.

It is clear that if the middle support were taken away, the weight of the beam would cause it to bend down at the middle [as found above by equation (X)] through a distance

$$\Delta_1 = \frac{5 \ w \ L^4}{384 \ E \ I} \text{ inches.}$$

We have also seen by equation (XII) that a single concentrated load of W lbs. applied at the middle of the beam would produce an amount of deflection, $\Delta_2 = \dfrac{W \ L^3}{48 \ E \ I}$ inches.

This gives us the upward deflection caused by the reaction of the central support if we put its value, P, instead of W in the equation.

The total deflection will be zero if all three supports are on the same level.

Then, $\dfrac{P \ L^3}{48 \ E \ I} = \dfrac{5 \ w \ L^4}{384 \ E \ I}.$

Or, $P = \tfrac{5}{8} \ w \ L.$

The pressure on the middle support is thus seen to be $\tfrac{5}{8}$ of the weight of the beam ; whilst the end supports each carry $\tfrac{3}{16}$ of the weight.

Uniform Beam Fixed at one End and Supported at the other.— If w be the weight of the beam in lbs. per inch-run and L its length in inches, then, we already know that at x inches from the fixed end, the

$$\text{B.M.} = \tfrac{1}{2} \ w \ (L - x)^2 \text{ inch-lbs.}$$

Putting this value of the B.M. in equation (VIII), we get :—

$$\frac{d^2 y}{d x^2} = \frac{w}{2 \ E \ I} (L - x)^2 = \frac{w}{2 \ E \ I} (L^2 - 2 \ L \ x + x^2).$$

$$\therefore \quad \frac{d y}{d x} = \frac{w}{2 \ E \ I} \int (L^2 - 2 \ L \ x + x^2) \, d x$$

$$,, \ = \frac{w}{2 \ E \ I} \{ L^2 \ x - L \ x^2 + \tfrac{1}{3} \ x^3 \}.$$

And, $y = \dfrac{w}{2 \ E \ I} \displaystyle\int \{ L^2 \ x - L \ x^2 + \tfrac{1}{3} \ x^3 \} d x$

$$y = \frac{w}{2 \ E \ I} \{ \tfrac{1}{2} L^2 x^2 - \tfrac{1}{3} L x^3 + \tfrac{1}{12} x^4 \}. \tag{XIII}$$

This last equation gives the droop of the beam at any distance x, from the fixed end. At the free end let Δ_1 be the value of y when $x = L$.

Then, $$\Delta_1 = \frac{w\,L^4}{8\,E\,I} \text{ inches.} \quad . \quad . \quad . \quad . \quad \text{(XIV)}$$

Let P be the upward pressure in lbs., between the beam and a support placed under its free end. The bending moment, due to P, at x inches from the fixed end is $P(L - x)$ inch-lbs. Hence, the curvature produced by P will be :—

$$\frac{d^2 y}{d x^2} = \frac{P}{E\,I}(L - x).$$

$$\therefore \quad \frac{dy}{dx} = \frac{P}{E\,I}\int(L - x)\,d x = \frac{P}{E\,I}(L\,x - \tfrac{1}{2}x^2).$$

And, $$y = \frac{P}{E\,I}\int(L\,x - \tfrac{1}{2}x^2)\,d x = \frac{P}{2\,E\,I}(L\,x^2 - \tfrac{1}{3}x^3). \quad . \quad \text{(XV)}$$

When $x = L$, let $y = \Delta_2$.

$$\therefore \quad \Delta_2 = \frac{P\,L^3}{3\,E\,I} \text{ inches.} \quad . \quad . \quad . \quad . \quad \text{(XVI)}$$

If $\Delta_2 \quad \Delta_1$, the supported end will be raised to the same level as the fixed end.

Then, $$\frac{P\,L^3}{3\,E\,I} = \frac{w\,L^4}{8\,E\,I}.$$

Or, $$P = \tfrac{3}{8}\,w\,L \text{ lbs.}$$

This result shows that the pressure on the prop is equal to $\frac{3}{8}$ of the weight of the beam.

It will be instructive for the student to observe that this result might easily have been inferred from the previous case of a beam resting on three props.

In that case the part of the beam immediately over the middle support is in exactly the same condition as the fixed end of the beam in this case ; so that whatever is true of each half of the beam in the former case will here hold good for the whole beam. The pressure on the end supports is, therefore, identical in magnitude in each case ; because $\frac{3}{16}$ of the weight of the whole beam is the same thing as $\frac{3}{8}$ of the weight of each half.

Beam Fixed at Both Ends and Loaded at the Centre.—When a beam is fixed, or built horizontally into a wall at both ends, the fixing causes a bending moment which is constant all over the

beam. For the reaction of the left support in keeping the beam horizontal is equivalent to a force P, acting downwards at some distance l, to the left of that support, and an upward force P, at the support. The bending moment at the support is then :—

$$M_s = P \times l.$$

And, at any other point, E, of the beam, at a distance, z (less than half the span from the support), the B.M. caused by this reaction at the support is :—

$$B.M. = P(z + l) - Pz = Pl = M_s.$$

Consequently, the fixing at the ends causes a constant B.M. all over the beam, equal to that at the supports, in addition to that caused by the load (*but in the opposite direction*).
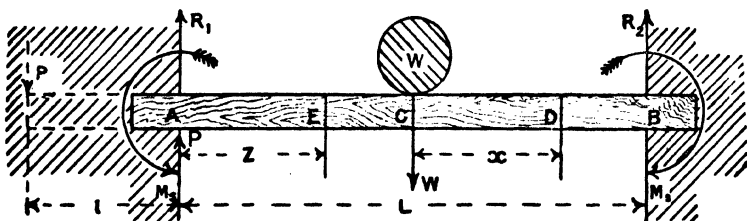
BEAM FIXED AT ENDS AND LOADED AT CENTRE.

Taking our origin of co-ordinates at C, the centre, and the undeflected axis, or neutral line of the beam, as our axis of x, we have, at a section D, distant x from C :—

$$B.M. = R_2(\tfrac{1}{2}L - x) - M_s = \tfrac{1}{2}W(\tfrac{1}{2}L - x) - M_s$$

Hence, from eqn. (VIII), $\quad \dfrac{d^2y}{dx^2} = \dfrac{1}{EI} \{ \tfrac{1}{2}W(\tfrac{1}{2}L - x) - M_s \}$

$$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{EI} \{ \tfrac{1}{4}W(Lx - x^2) - M_s x \}$$

The beam is horizontal at the centre and at the ends, therefore $\dfrac{dy}{dx}$ is zero when x is zero, and when $x = \tfrac{1}{2}L$

$$\therefore \quad 0 = \dfrac{1}{EI} \{ \tfrac{1}{4}W (\tfrac{1}{2}L^2 - \tfrac{1}{4}L^2) - \tfrac{1}{2}M_s L \}$$

$$\therefore \quad M_s = \tfrac{1}{8} W L$$

Inserting this value in the above equation for the B.M. we get :—

$$\text{B.M.} = \tfrac{1}{2}W\left(\tfrac{1}{2}L - x\right) - \tfrac{1}{8}W L$$

Or, $$\text{B.M.} = \tfrac{1}{2}W\left(\tfrac{1}{4}L - x\right) \quad . \quad . \quad . \quad (XVII)$$

At the centre, $$\text{B.M.} = \tfrac{1}{2}W \cdot \tfrac{1}{4}L = \tfrac{1}{8}W L = M_s$$

$\therefore$ Maximum B.M. $= M_s = \tfrac{1}{8}W L. \quad . \quad . \quad . \quad (XVIII)$
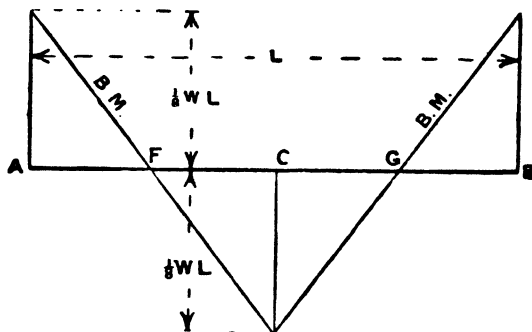
B.M. Diagram for Beam Fixed at Ends and Loaded at Centre.

We thus see that, in this case the fixing of the ends reduces the maximum B.M. to half what it would be with free ends, and that this maximum B.M. occurs both at the centre and the ends.

The B.M. diagram is similar to what we had for a beam simply supported, but the base line is shifted half way down the diagram, so that it is crossed at F and G by the lines representing the B.M. It will be seen from equation (XVII) that the B.M. is zero where $x = \tfrac{1}{4} L$, and that it is positive on one side of this point, and negative on the other. This is one of the points where the B.M. curve cuts the base line, and it is called a *point of inflection* or *point of contraflexure*, because the beam is straight just at that point and the curvature changes sign. There is, of course, another point of inflection at the distance $\tfrac{1}{4} L$ on the other side of the centre.

In large girder bridges that part of the span between the two points of inflection is made separate from the remainder and rests on rollers at these points. This allows freedom of expansion without reducing the strength of the bridge.

Integrating the value of $\dfrac{d y}{d x}$, we get :—

$$y = \frac{1}{E I}\left\{ \tfrac{1}{4} W \left(\tfrac{1}{2} L x^2 - \tfrac{1}{3} x^3\right) - \tfrac{1}{16} W L x^2\right\}$$

Therefore, at the ends, where $x = \frac{1}{2} L$:—

$$y = \frac{1}{EI} \left\{ \frac{1}{4} W \left(\frac{L^3}{8} - \frac{L^3}{24} \right) - \frac{1}{16} W \frac{L^3}{4} \right\} = \frac{W L^3}{192 \, EI}.$$

Hence, the difference of level between the centre and the ends is :—

$$\Delta = \frac{W L^3}{192 \, EI} \quad \dots \dots \quad \text{(XIX)}$$

This is only one-fourth of the deflection when the beam simply rested on its supports (Equation XII), so that the beam is now four times as stiff.

Beam Fixed at Both Ends and Loaded Uniformly.—Taking axes as before, the B.M. at any section, D, is :—

$$\text{B.M.} = R_2 \left(\tfrac{1}{2} L - x \right) - \tfrac{1}{2} w \left(\tfrac{1}{2} L - x \right)^2 - M_s.$$

Or, $\text{B.M.} = \tfrac{1}{2} w L \left(\tfrac{1}{2} L - x \right) - \tfrac{1}{2} w \left(\tfrac{1}{4} L^2 - L x + x^2 \right) - M_s.$

$$\therefore \quad \frac{d^2 y}{d x^2} = \frac{1}{EI} \left\{ \tfrac{1}{2} w \left(\tfrac{1}{4} L^2 - x^2 \right) - M_s \right\}.$$

Or, $\dfrac{d y}{d x} = \dfrac{1}{EI} \left\{ \tfrac{1}{2} w \left(\tfrac{1}{4} L^2 x - \tfrac{1}{3} x^3 \right) - M_s x \right\}.$
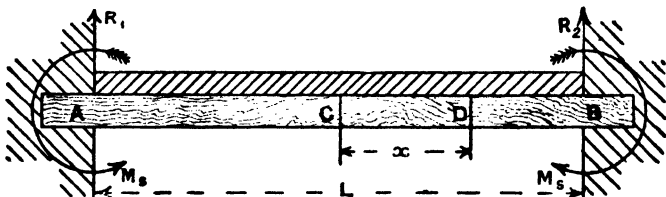
BEAM FIXED AT BOTH ENDS AND LOADED UNIFORMLY.

In this case also, $\dfrac{d y}{d x}$ is zero when x is zero, and when $x = \tfrac{1}{2} L.$

$$\therefore \quad 0 = \tfrac{1}{2} w L^3 \left(\tfrac{1}{8} - \tfrac{1}{24} \right) - \tfrac{1}{2} M_s L.$$

Or, $\quad M_s = \dfrac{w L^2}{12} = \dfrac{W L}{12}.$

Hence, $\quad \text{B.M.} = \tfrac{1}{2} w \left(\tfrac{1}{4} L^2 - x^2 \right) - \tfrac{1}{12} w L^2.$

Or, $\quad \text{B.M.} = \tfrac{1}{2} w \left(\tfrac{1}{12} L^2 - x^2 \right). \quad \dots \dots \quad \text{(XX)}$

At the centre, where $x = 0$. the B.M. is :—

$$M_c = \frac{w L^2}{24} = \frac{W L}{24} = \tfrac{1}{2} M_s$$

This is only half of that at the support. Hence, the greatest bending moment occurs at the support, and its value is:—

$$\text{Maximum B.M.} = \tfrac{1}{12} w \, L^2 = \tfrac{1}{12} \, W \, L. \quad . \quad . \text{(XXI)}$$
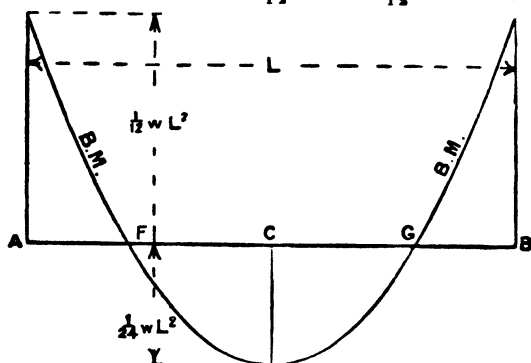
B.M. DIAGRAM FOR BEAM FIXED AT BOTH ENDS AND LOADED UNIFORMLY.

The points of inflection occur where $x^2 = \tfrac{1}{12} L^2$ or $x = \pm \dfrac{L}{2\sqrt{3}}$.

By integrating the above value of $\dfrac{d\,y}{d\,x}$:—

$$y = \frac{1}{EI} \left\{ \tfrac{1}{2} w \left(\tfrac{1}{8} L^2 x^2 - \tfrac{1}{12} x^4 \right) - \tfrac{1}{12} w \, L^2 \frac{x^2}{2} \right\} = \frac{w}{48\,E\,I} (L^2 x^2 - 2\,x^4)$$

Putting $x = \dfrac{L}{2}$ we obtain the amount by which the centre of the beam is deflected by the load, viz.:—

$$\Delta = \frac{w}{48\,E\,I} \left(\frac{L^4}{4} - \frac{L^4}{8} \right) = \frac{w\,L^4}{384\,E\,I} = \frac{W\,L^3}{384\,E\,I} \quad . \quad . \text{(XXII)}$$

We thus see that, by fixing the ends horizontally for this manner of loading, the strength of the beam is increased in the ratio 3 : 2, and its stiffness in the ratio 5 : 1.

When the span of the beam is small, it may be designed wholly from considerations of strength; but when the span is great a beam may be strong enough, and yet not suitable, because it yields too much when the load is put on it. It then becomes necessary to take the stiffness into account by using one of the formulæ we have found for the deflection. The greatest deflection usually allowed in beams is 1 inch in 100 feet, or $\tfrac{1}{1200}$ of the span.

In the tables below we give a summary of these results, showing the relation between them.

TABLE III.—STRENGTH AND STIFFNESS OF BEAMS UNDER A TOTAL LOAD OF W LBS.
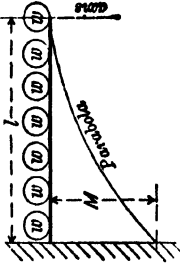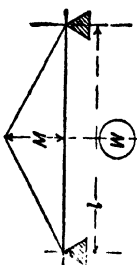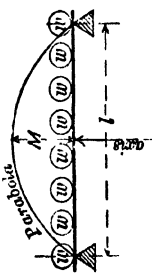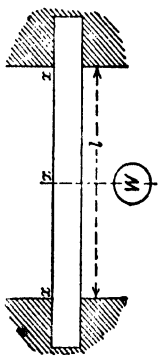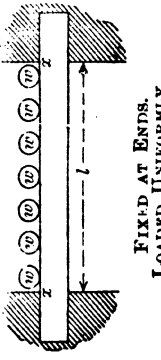
No.	Manner of Supporting and Loading	Maximum Bending Moment = M	Relative Strength	Deflection in Terms of W.	Deflection in Terms of M	Deflection in Terms of Stress	Relative Stiffness under same Load.
I.	CANTILEVER LOADED AT END.	$W l$	$\frac{1}{8}$	$\frac{1}{3} \cdot \frac{W l^3}{EI}$	$\frac{1}{3} \cdot \frac{M l^2}{EI}$	$\frac{1}{3} \cdot \frac{f l^2}{Ey}$	$\frac{1}{16}$
II.	CANTILEVER LOADED UNIFORMLY.	$\dfrac{W l}{2}$	$\frac{1}{4}$	$\frac{1}{8} \cdot \frac{W l^3}{EI}$	$\frac{1}{4} \cdot \frac{M l^2}{EI}$	$\frac{1}{4} \cdot \frac{f l^2}{Ey}$	$\frac{1}{6}$
III.	SUPPORTED AT BOTH ENDS. LOADED AT CENTRE.	$\dfrac{W l}{4}$	1	$\frac{1}{48} \cdot \frac{W l^3}{EI}$	$\frac{1}{12} \cdot \frac{M l^2}{EI}$	$\frac{1}{12} \cdot \frac{f l^2}{Ey}$	1

TABLE III.—STRENGTH AND STIFFNESS OF BEAMS UNDER A TOTAL LOAD OF W LBS. (continued).

No.	MANNER OF SUPPORTING AND LOADING.	Maximum Bending Moment = M	Relative Strength.	Deflection in Terms of W.	Deflection in Terms of M.	Deflection in Terms of Stress.	Relative Stiffness under same Load.
IV.	SUPPORTED AT BOTH ENDS. LOADED UNIFORMLY.	$\dfrac{W\,l}{8}$	2	$\dfrac{5}{384}\cdot\dfrac{W\,l^3}{EI}$	$\dfrac{5}{48}\cdot\dfrac{M\,l^2}{EI}$	$\dfrac{5}{48}\cdot\dfrac{f\,l^2}{Ey}$	4
V.	FIXED AT ENDS. LOADED AT CENTRE.	$\dfrac{W\,l}{8}$	2	$\dfrac{1}{192}\cdot\dfrac{W\,l^3}{EI}$	$\dfrac{1}{24}\cdot\dfrac{M\,l^2}{EI}$	$\dfrac{1}{24}\cdot\dfrac{f\,l^2}{Ey}$	4
VI.	FIXED AT ENDS. LOADED UNIFORMLY.	$\dfrac{W\,l}{12}$	3	$\dfrac{1}{384}\cdot\dfrac{W\,l^3}{EJ}$	$\dfrac{1}{32}\cdot\dfrac{M\,l^2}{EI}$	$\dfrac{1}{16}\cdot\dfrac{f\,l^2}{Ey}$	8

The quantities in the sixth column are obtained by substituting the value of the maximum B.M. given by the third column in the fifth, and for those in the seventh we have put the value of M $\left(\text{viz., } \dfrac{f\,I}{y}\right)$ found in equation (V).

We also print for reference a table of the strengths of materials and of the moduli of different sections.

For properties of British Standard Sections, see Appendix.

BIBLIOGRAPHY.

The following books may be consulted by those students who wish to extend their studies to a more advanced stage :—

Materials of Construction, by Johnson. (Chapman & Hall, Ltd.)
Strength of Materials, by Sir J. Ewing. (Cambridge University Press.)
Strength of Materials, by A. Morley. (Longmans, Green & Co.)
Strength of Materials, by Ewart S. Andrews. (Chapman & Hall, Ltd.)
Strength of Materials, by Popplewell. (Scientific Publishing Co.)
Testing of Materials, by Unwin. (Longmans, Green & Co.)
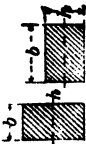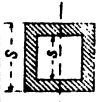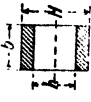
STRENGTHS, &C., OF MATERIALS (SUMMARY).
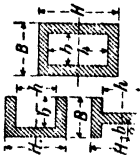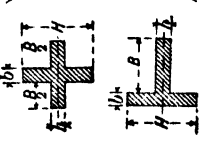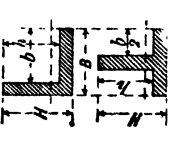
Material.	Ultimate Tensile Strength. lbs. per square inch.	Elastic Strength. lbs. per square inch.	Elongation per cent., when broken by Tensile Stress.
Cast-iron (ordinary good)	18,000	11,000	...
,, (Admiralty), .	{ not less than 20,160 }	...	...
Wrought-iron bars (ordinary good), . .	54,000	29,000	15 °/₀ in 8 ins.
Yorkshire plate—			
With grain, . .	54,000	26,000	20 °/₀ ,,
Across ,, . .	49,000	...	14 °/₀ ,,
Staffordshire plate—			
With grain, . .	50,000	24,000	12 °/₀ ,,
Across ,, . .	41,000	...	8 °/₀ ,,
Iron forgings—			
Large, . . .	45,000	...	9 °/₀ ,,
Small, . . .	50,000	...	13 °/₀ ,,
Steel castings (ordinary good), . . .	67,000	35,000	10 °/₀ ,,
Steel castings (Admiralty)	{ not less than 63,000 }	...	{ not less than 13½–18½ °/₀ in 2 ins.
,, (Lloyd's), .	{ not exceeding 67,000 }	...	{ not less than 10 °/₀ in 8 ins.
Steel boiler plate—			
(Ordinary good), .	65,000	36,000	20 °/₀ ,,
(Admiralty) internal,	{ not exceeding 60,480 }	...	20 °/₀ ,,
,, shell, .	60,480-67,200	{ not less than 31,360	...
(B. of T.) internal, .	58,240-67,200	...	...
,, shell, .	60,480–71,680	...	18 °/₀ in 10 ins.
Lloyd's, . . .	58,240-67,200	...	{ not less than 20 °/₀ in 8 ins.
Steel forgings (Admiralty)	62,720–78,400	{ 34,500 to 43,120	{ 28 °/₀ to 24 °/₀ in 2 ins. }
Sheet copper, . .	30,000	5,600	35 °/₀ in 8 ins.
Copper wire (annealed),	40,000	...	...
Gun - metal (ordinary good). . . .	27,000	6,500	10 °/₀ in 2 ins.
Gun-metal (Admiralty),	31,000	...	...
Phosphor bronze (cast),	35,000	19,000	12 °/₀ in 2 ins.
Manganese bronze ,,	55,000	...	10 °/₀ ,,
,, (rolled),	67,000	...	20 °/₀ ,,
Muntz metal, . .	50,000	30,000	30 °/₀ ,,
Naval brass, . .	54,000	24,000	25 °/₀ in 8 ins.

MOMENT OF INERTIA, MODULUS, &c., OF SOME SECTIONS.

The plane of bending is supposed perpendicular to plane of paper, and parallel to side of page.

Form of Section.	Area of Section. A	Moment of Inertia of Section about Axis through Centre of Gravity. I	Square of rads. of gyration of Section. $\rho^2 = \dfrac{I}{A}$	"Modulus" of Section. $Z = \dfrac{I}{y}$
(square section)	a^2	$\dfrac{a^4}{12}$	$\dfrac{a^2}{12}$	$\dfrac{a^3}{6}$
(rectangular section)	bh	$\dfrac{bh^3}{12}$	$\dfrac{h^2}{12}$	$\dfrac{bh^2}{6}$
(hollow square section)	$S^2 - s^2$	$\dfrac{S^4 - s^4}{12}$	$\dfrac{S^2 + s^2}{12}$	$\tfrac{1}{6}\left(\dfrac{S^4 - s^4}{S}\right)$
(hollow rectangular section)	$b(H - h)$	$\dfrac{b(H^3 - h^3)}{12}$	$\dfrac{H^2 + hH + h^2}{12}$	$\dfrac{b(H^3 - h^3)}{6H}$
(circular section)	$\cdot 7854\,d^2$	$\cdot 0491\,d^4$	$\dfrac{d^2}{16}$	$\cdot 0982\,d^3$
(hollow circular section)	$\cdot 7854\,(D^2 - d^2)$	$\cdot 0491\,(D^4 - d^4)$	$\dfrac{D^2 + d^2}{16}$	$\cdot 0982\left(\dfrac{D^4 - d^4}{D}\right)$

MOMENT OF INERTIA, MODULUS, &c., OF SOME SECTIONS—*Continued.*

Form of Section.	A as of Section. A	Moment of Inertia of Section about Axis through Centre of Gravity. I	Square of rads. of gyration of Section. $\rho^2 = \dfrac{I}{A}$	"Modulus" of Section $z = \dfrac{I}{y}$
	$BH - bh$	$\dfrac{BH^3 - bh^3}{12}$	$\dfrac{1}{12}\left(\dfrac{BH^3 - bh^3}{BH - bh}\right)$	$\dfrac{BH^3 - bh^3}{6H}$
	$Bh + bH$	$\dfrac{Bh^3 + bH^3}{12}$	...	$\dfrac{Bh^3 + bH^3}{6H}$
	$BH - bh$	$\dfrac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{12(BH - bh)}$	...	$\dfrac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{6(BH^2 + bh^2 - 2bHh)}$

EXAMPLE VII.—A rigid body of unit width is rectangular in vertical section, which is 30 feet long and 10 feet high. A force of 20 tons is applied vertically to its upper surface, one-third the length from one vertical face, on the centre of which a normal force of 10 tons acts. If the specific gravity of the block is $2\frac{1}{4}$, find the distribution of pressure on the base.

ANSWER.—
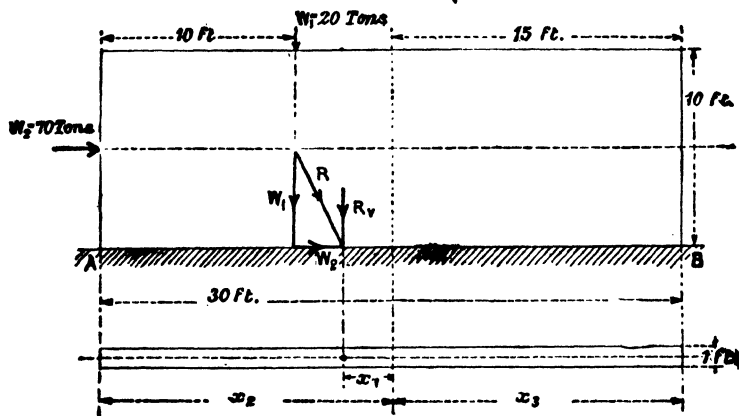
LINE DIAGRAM OF LOADED RIGID BODY, THE LOAD BEING NON-AXIAL.

In this question we may consider it as made up of two parts :—

(1) The weight of the body itself may be taken as giving a *uniform* stress all over the section or base.

(2) Compounding by the "parallelogram of forces" the two external forces W_1 and W_2, so as to give a resultant R, which cuts the base at a point $2\frac{1}{2}$ feet from the centre of area. The vertical component R_v of the resultant is therefore the load acting vertically downwards, while the horizontal component tends to produce sliding of the body along the ground.

(1) *Uniform Stress.*—Since the weight of the body is acting along the axis of the body, it produces a uniform distribution of stress over the surface of a cross-section; the intensity of stress at all points of the surface is uniform and constant; and the resultant of the stress on the surface acts at a point called the *centre of stress*, which, in this part, coincides with the centre of area.

$$\text{Weight of body} = \frac{1 \times 10 \times 30 \times 62 \cdot 5 \times 9}{2 \cdot 240 \times 4} = 18 \cdot 83 \text{ tons.}$$

$\therefore$ Uniform compressive stress on base due to *weight of body only* $\Big\}$ = $\dfrac{\text{weight of body}}{\text{cross-sectional area of base}}$

,, ,, = $\dfrac{18 \cdot 83}{30}$ = $\cdot 628$ ton per sq. ft.

(2) *Uniformly-varying Stress.*—Here the centre of stress of the cross-section does not coincide with the centre of area; then the distribution of stress over the surface is unequal, and it is assumed that the stress is a *uniformly varying* one—that is, the intensity of stress at any point in the section varies directly as the distance of that point from a fixed line in the plane of the section.

SKETCH SHOWING STRESS ON A SECTION OR JOINT, THE LOAD BEING NON-AXIAL.

Let A B = Trace of the base on a plane at right angles to it.
,, C = End of a line through its centre of area.
,, R_v = Component of the resultant R normal to the surface.
,, x_1 = Distance of centre of stress from centre of area.
,, x_2, x_3 = Distances from C of B and A respectively.
,, A_s = Area of surface or base A B.
,, f_1, f_2 = Extreme intensities of stress at A and B.
,, f_c = Intensity of stress at centre of area of A B.
,, I = Moment of inertia of the surface about an axis through C at right angles to the plane of the figure.

Then the stress represented by the rectangle may be considered as made up of two parts, viz. :—

(*a*) A uniform stress due to a load equal to R_v acting at the centre of area C, the intensity of which is $f_c = \dfrac{R_v}{A_s}$; and

(*b*) A uniformly-varying stress due to a bending moment (B M) = $R_v x_1$.

The intensity of this uniformly-varying stress on any line distant x from O is :—

$$\frac{(B\,M)\,x}{I} = \frac{R_v\,x_1\,x}{I}.$$

Adding (a) and (b), we get :—

The intensity of stress at edge A,

$$f_1 = f_o + \frac{R_v\,x_1\,x}{I} = \frac{R_v}{A_s}\left(1 + \frac{A_s\,x_1\,x}{I}\right). \quad . \quad . \quad . \quad . \quad (1)$$

The intensity of stress at edge B,

$$f_2 = f_o - \frac{R_v\,x_1\,x}{I} = \frac{R_v}{A_s}\left(1 - \frac{A_s\,x_1\,x}{I}\right). \quad . \quad . \quad . \quad . \quad (2)$$

Since the section is symmetrical, $x_2 = x_3$; and equations (1) and (2) become :—

$$f_1 = \frac{R_v}{A_s}\left(1 + \frac{A_s\,x_1\,x_2}{I}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$f_2 = \frac{R_v}{A_s}\left(1 - \frac{A_s\,x_1\,x_2}{I}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In this question :—

$$A_s = 30 \text{ square feet,}$$

$$I = \frac{b\,d^3}{12} = \frac{1 \times (30)^3}{12} = \frac{27,000}{12} = 2,250 \text{ (feet)}^4 \text{ units,}$$

$$x_1 = 2\tfrac{1}{2} \text{ feet,} \qquad x_2 = 15 \text{ feet.}$$

$$\therefore \qquad f_1 = \frac{R_v}{A_s}\left(1 + \frac{A_s\,x_1\,x_2}{I}\right),$$

$$f_1 = \frac{20}{30}\left(1 + \frac{30 \times 5 \times 15}{2 \times 2,250}\right)$$

$$f_1 = \tfrac{2}{3}(1 + \tfrac{1}{2}) = 1 \text{ ton per square foot (compressive).}$$

$$f_2 = \tfrac{2}{3}(1 - \tfrac{1}{2}) = \tfrac{1}{3} \text{ ton per square foot (compressive).}$$

To obtain the maximum values of the intensities of the stress at the ends A and B, we must add the value of the uniform compressive stress due to the weight of the *body only*, to *each* of the above values, f_1 and f_2.

∴. Maximum compressive stress at A

$$= 1 + 0·628 = 1·628 \text{ tons per square foot.}$$

Minimum compressive stress at B

$$= 0·3 + 0·628 = 0·961 \text{ ton per square foot.}$$

Thick Pipes.—We have considered already the strength of a thin pipe and obtained simple formulæ by assuming that the stress was constant throughout the length and thickness of the pipe. When a pipe is not very thin compared with its diameter we have to allow for the variation of stress across the section.

Lamé's Theory.—Let a pipe be of internal radius r and external radius R, and let it be under pressure either from the inside or from the outside.

Now consider an imaginary thin ring of thickness δx and internal radius x. This ring will be subjected to a radial pressure p on the inside which by considerations of symmetry must be the same all round, and on the outside it will be subjected to a radial pressure which will differ slightly from p, and which we may call $p + \delta p$. This assumes that the tube is subjected to pressure on the outside; if it is on the inside the same formulæ hold with appropriate change of sign, as explained later.

We may therefore apply to this imaginary hoop the same treatment as for a thin pipe, the circumferential stress, or *hoop stress*, being f.

Considering a unit length of pipe we have

Force tending to cause collapse of ring $= (p + \delta p) \times 2 (x + \delta x)$.

Force resisting collapse of ring $= 2f \, \delta x + p \, x \,.\, 2 \, x.$

These must be equal.

∴ dividing by 2 and neglecting the product, $\delta p \,.\, \delta x$, of two very small quantities, we have

$$p \,.\, x + x \,.\, \delta p + p \, \delta x = f \, \delta x + p \,.\, x.$$

∴
$$(f - p) \, \delta x = x \, \delta p$$

$$(f - p) = \frac{x \, \delta p}{\delta x}.$$

In the limit when the increments are infinitely small this gives

$$(f - p) = \frac{x \, d p}{d x}. \quad . \quad . \quad . \quad . \quad (1)$$

This is one relation between f and p.

Now let us assume that the strains along the length of the pipe will be such that a plane section before subjection to

2   6

pressure remains plane after subjection to pressure—*i.e.*, that longitudinal strain is constant.

i.e.,

$$\frac{\eta f}{E} + \frac{\eta p}{E} = \text{constant}, \quad . \quad . \quad . \quad (2)$$

because both f and p will cause transverse strains in the direction of the length of the pipe, and they will have the same sign.

∴ Since η and E are constant, if our stresses are within the elastic limit we may write

$$f + p = \text{constant} = 2\,a \text{ (say)}.$$

∴

$$f = (2\,a - p). \quad . \quad . \quad . \quad . \quad (3)$$
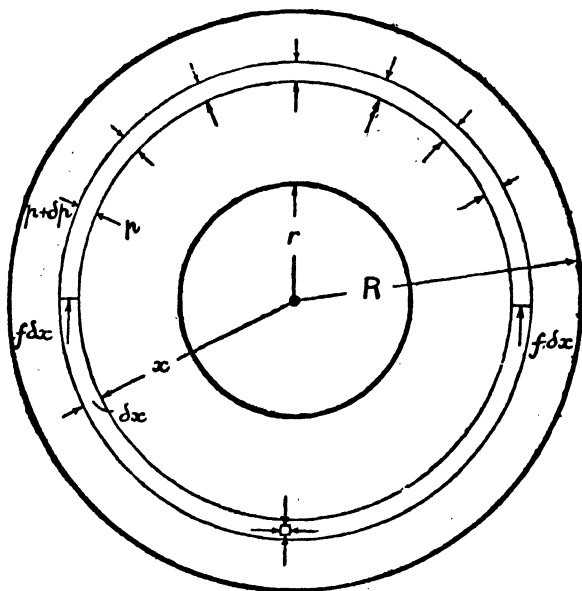
STRESSES IN THICK PIPES.

Put this value in (1) and we get

$$2\,a - 2\,p = \frac{x\,d\,p}{d\,x}$$

$$2\,a = 2\,p + \frac{x\,d\,p}{d\,x}; \quad . \quad . \quad . \quad (4)$$

but
$$\frac{d\,(p\,x^2)}{d\,x} = 2\,p\,x + \frac{x^2\,d\,p}{d\,x}$$

$$= x\left(2\,p + \frac{x\,d\,p}{d\,x}\right) = 2\,a\,x.$$

$\therefore$
$$d\,(p\,x^2) = 2\,a\,x\,.\,d\,x. \qquad . \qquad . \qquad . \qquad . \quad (5)$$

Integrating we get
$$p\,x^2 = a\,x^2 + b,$$

where b is a constant.

$\therefore$
$$p = a + \frac{b}{x^2}; \quad . \qquad . \qquad . \qquad . \qquad . \quad (6)$$

but
$$f = 2\,a - p \text{ (by 3)};$$

i.e.,
$$f = a - \frac{b}{x^2}. \quad . \qquad . \qquad . \qquad . \quad (7)$$

By calculating a and b in any particular case we can find formula for p and f.

Special Case. — (1) PRESSURE INSIDE $= p_i$; PRESSURE OUTSIDE $= 0$,

i.e.,
$$p = p_i \text{ for } x = r,$$

$$p = 0 \text{ for } x = R;$$

$\therefore$
$$p_i = a + \frac{b}{r^2}. \qquad . \qquad . \qquad . \qquad . \quad (8)$$

$$0 = a + \frac{b}{R^2};$$

$\therefore$
$$a = -\frac{b}{R^2}.$$

Put this value in (8), then
$$p_i = b\left(\frac{1}{r^2} - \frac{1}{R^2}\right) = \frac{b\,(R^2 - r^2)}{R^2\,r^2};$$

$\therefore$
$$b = \frac{p_i\,R^2\,r^2}{R^2 - r^2} \qquad . \qquad . \qquad . \qquad . \quad (9)$$

$\therefore$
$$a = -\frac{b}{R^2} = -\frac{p_i\,r^2}{R^2 - r^2} \qquad . \qquad . \quad (10)$$

∴ *Hoop stress at inside* $= f_i$ is obtained by putting $x = r$ in (7),

i.e.,
$$f_i = a - \frac{b}{r^2}$$

$$= -\frac{p_i r^2}{R^2 - r^2} - \frac{p_i R^2}{R^2 - r^2}$$

$$= -\frac{p_i (R^2 + r^2)}{R^2 - r^2}. \qquad . \qquad . \qquad . \qquad . \quad (11)$$

The negative sign indicates that the stress is a tension.

Hoop stress at outside $= f_o = a - \dfrac{b}{R^2}$

$$= -\frac{p_i r^2}{(R^2 - r^2)} - \frac{p_i r^2}{(R^2 - r^2)}$$

$$= -\frac{2 p_i r^2}{(R^2 - r^2)}. \qquad . \qquad . \qquad . \qquad . \quad (12)$$

This is also a tensile stress, and is clearly less than f_i, so that with internal pressure the maximum stress occurs on the inside.

At any intermediate radius x

$$p = a + \frac{b}{x^2}$$

$$= -\frac{p_i r^2}{(R^2 - r^2)} + \frac{p_i R^2 r^2}{(R^2 - r^2) x^2}$$

$$= \frac{p_i r^2}{(R^2 - r^2)} \left\{ \frac{R^2}{x^2} - 1 \right\}. \qquad . \qquad . \qquad . \quad (13)$$

$$f = a - \frac{b}{x^2}$$

$$= -\frac{p_i r^2}{(R^2 - r^2)} \left\{ \frac{R^2}{x^2} + 1 \right\}. \qquad . \qquad . \quad (14)$$

It should be noted from equation (11) that no matter how great the thickness of the tube may be, the hoop stress is always greater than the internal pressure, so that for any given material there is a certain maximum pressure which must not be exceeded.

It should also be noted that the assumption of the constancy of longitudinal strain holds only while the stress is within the elastic limit.

NUMERICAL EXAMPLES.—(1) *A cast-steel cylinder 2 ft. in external diameter and 3 inches thick is subjected to an internal pres-*

sure of 2 *tons per sq. in. Where and of what magnitude is the maximum stress?*

The maximum stress is on the inside, and is given by the formula—

$$f_i = \frac{p\,(R^2 + r^2)}{(R^2 - r^2)}$$

$$= \frac{2\,(24^2 + 18^2)}{(24^2 - 18^2)} = \frac{2 \times 6^2\,(16 + 9)}{6^2\,(16 - 9)}$$

$$= \frac{2 \times 25}{7} = 7\cdot14 \text{ tons per sq. in.}$$
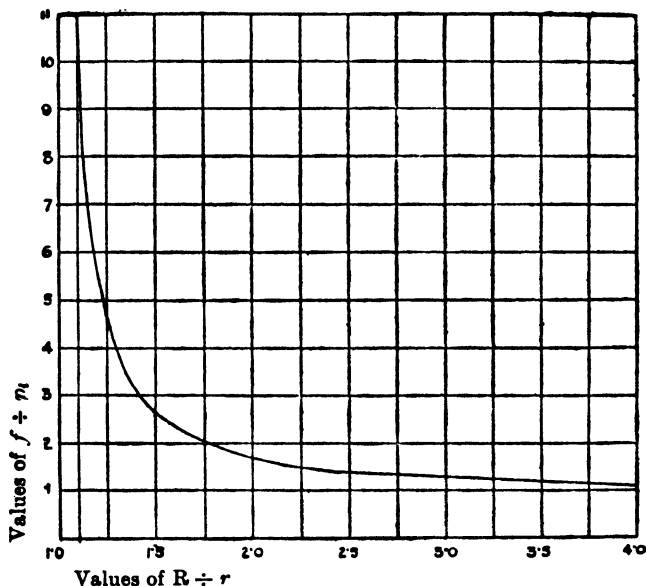
VARIATION OF HOOP STRESS FOR VARIOUS RATIOS OF EXTERNAL TO INTERNAL RADII OF PIPES WITH INTERNAL PRESSURE.

(2) *Plot a curve showing the maximum stress in terms of the internal pressure in a tube whose ratio of external to internal radius varies from* 1·10 *to* 4.

$$f_i = \frac{p_i\,(R^2 + r^2)}{(R^2 - r^2)} \; ;$$

$$\therefore \qquad \frac{f_i}{p_i} = \frac{R^2 + r^2}{R^2 - r^2} = \frac{\left(\dfrac{R}{r}\right)^2 + 1}{\left(\dfrac{R}{r}\right)^2 - 1} \, .$$

This gives the following values :—

$\dfrac{R}{r}$	1·10	1·20	1·30	1·50	2·00	2·50	3·00	3·50	4·00
$\dfrac{f_i}{p_i}$	10·52	5·55	3·90	2·60	1·67	1·38	1·25	1·18	1·13
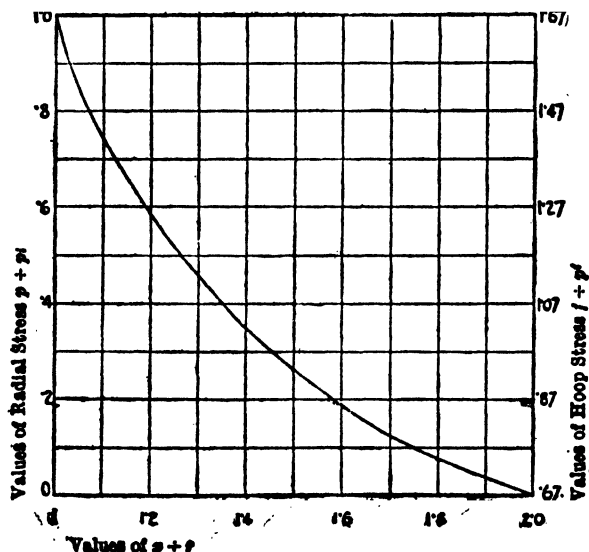
VARIATION OF STRESS IN A THICK PIPE WITH INTERNAL PRESSURE

If $\dfrac{R}{r} = 1\cdot10$, $\dfrac{r+t}{r} = 1\cdot10$, $\dfrac{t}{r} = \cdot10$.

$\therefore$ The thin pipe formula $f = \dfrac{pr}{t}$ would give $\dfrac{f}{p} = 10$, so that the thin pipe formula would be about 5 per cent. in error.

The above figures give the curve.

Curves of Variation of Radial and Hoop Stress for Internal Pressure for $R = 2\,r$.

By equation (9) $\dfrac{b}{p_i} = \dfrac{R^2\,r^2}{R^2 - r^2} = \dfrac{4\,r^4}{3\,r^2} = \dfrac{4\,r^2}{3}$.

By equation (10) $\dfrac{a}{p_i} = -\dfrac{r^2}{3\,r^2} = -\dfrac{1}{3}.$

$\therefore \qquad\qquad p = a + \dfrac{b}{x^2},$

$\dfrac{p}{p_i} = \dfrac{a}{p_i} + \dfrac{b}{p_i x^2} = -\dfrac{1}{3} + \dfrac{4\,r^2}{3\,x^2}$

$\qquad = \dfrac{4}{3}\left(\dfrac{r}{x}\right)^2 - \dfrac{1}{3} = \dfrac{1}{3}\left\{4\left(\dfrac{r}{x}\right)^2 - 1\right\}.$

$f = a - \dfrac{b}{x^2}.$

$\dfrac{f}{p_i} = -\left\{\dfrac{1}{3} + \dfrac{4}{3}\left(\dfrac{r}{x}\right)^2\right\} = -\dfrac{1}{3}\left\{4\left(\dfrac{r}{x}\right)^2 + 1\right\}.$

These results are plotted, and show clearly the variation of the stresses across the section.

1. A wrought-iron flanged girder is required to support a travelling load of 50 tons, the distance between the supports being 40 feet. The stress comes upon the girder at two points, the wheels on the traveller being 10 feet apart. What section of girder will be required to afford the necessary strength, presuming that the ultimate strength of the girder is six times that of the greatest stress to which it will be subjected?

2. Prove the law which governs the transverse strength of a beam of timber when supported at both ends and loaded at the centre. How are the constants required for applying this law arrived at?

3. A bar of wood, 7 feet long and 2 inches square, is supported at both ends, and is broken by a weight of 500 lbs. suspended at the centre. What weight in pounds will a rectangular bar of the same material, supported and loaded in like manner, sustain, when its length is 8 feet, its breadth 2½ inches, and its depth 4 inches? *Ans.* 2187·5 lbs.

4. A rectangular beam of fir, of uniform section throughout, is supported horizontally on two walls 15 feet apart, and has to carry a load of 1½ tons at 5 feet from one of the walls. The width of the beam is 5 inches; find its depth, taking the breaking load at four times the safe load. How much should the depth of the beam be increased, the breadth remaining constant, if the load were shifted from its original position to the centre of the beam, the breaking weight of a beam of fir 15 inches long, 1 inch broad, and 1 inch deep, supported at both ends and loaded in the middle, being taken at 360 lbs.? *Ans.* 8·9 inches; ½ inch.

5. A solid rectangular girder, 3 inches deep and 2 inches broad, is supported at both ends on supports 5 feet apart. It is loaded with a uniformly distributed load, including its own weight, of 10 cwts. per foot run. What is the maximum intensity of stress at the outer fibres? *Ans.* 14,000 lbs. per square inch.

6. If two cast-iron beams – one circular in section and 2·73 inches in diameter, the other of rectangular section, 3 inches broad and 2 inches deep—be each supported at two points 20 inches apart, and loaded at the centre with a load of 2 tons; what will be the maximum intensity of stress produced in each case? *Ans.* 5 tons per square inch in each case.

7. A beam of fir is built into a wall at one end, and projects 6 feet from the wall. The width of the beam is 4 inches; find its depth to bear safely a load of 1,200 lbs. uniformly distributed along its length. Assume that a bar of fir 1 foot long, 1 inch broad, and 1 inch deep, will break under a load of 125 lbs. when fixed at one end and loaded at the other end, and that the safe load is ¼ the breaking load. *Ans.* 6 inches.

8. What must be the breadth in inches of an oak cantilever or overhanging beam, 6 feet long and 9 inches deep, in order to carry a load of ⅓ ton at its extremity, and how much must its breadth be increased in order that it may carry an additional load of ½ ton uniformly distributed over its length? The actual stress is not to exceed ¼ of the breaking stress, and the breaking weight of an oak cantilever 6 inches long, 1 inch deep, and 1 inch broad, is 280 lbs. *Ans.* 2·37 inches; 1·18 inches.

9. A beam of fir supported at each end is inclined at an angle of 60° to the horizon, and is loaded at the centre of its length with a weight of 1 ton. The length of the beam is 10 feet, and its breadth is 2 inches, find the depth; the breaking load on the centre of a beam 1 foot long, 1 inch

broad, and 1 inch deep, and supported at the ends in a horizontal position, being 450 lbs.   *Ans.* 3·527 inches.

10. A cast-iron cantilever or overhanging beam of T-section is 6 feet long, and 9 inches deep, the top flange being 6 inches wide. The beam has to carry, with safety, at its end a load of 1 ton, together with a distributed load of 1 ton over its length. Find the thickness of the top flange, the tensile breaking strength of cast-iron being 8 tons per square inch, and the admissible load for a safe stress being one-fourth the breaking load. *Ans.* 1⅞ inches.

11. Find the greatest load that may be uniformly distributed on a cast-iron girder having top and bottom flanges united by a web of the following dimensions—width of upper flange 3 inches, of lower flange 9 inches, total depth 12 inches, thickness of each flange and of the web being 1 inch, distance between the points of support 10 feet—when the greatest admissible stress in the compression flange is 3 tons per square inch, and that in the tension flange is 1½ tons per square inch. *Ans.* Maximum compressive stress = 2·5 tons ; 8 5 tons.

12. Make a diagram of a flanged cast-iron girder to carry a load of 12 tons in the centre, the distance between the points of support being 20 feet. What should you make the depth of the beam, and what should be the sectional area of the top and bottom flanges respectively? *Ans.* 20·5 inches deep ; area top flange 7·5 sq. inches, of bottom flange 18·7 sq. inches.

13. A rolled steel girder has a mean depth of 10 inches, the top and bottom flanges are each 6 inches wide, and the metal in the flanges and webs is ½ inch in thickness throughout. If the breaking strength of the material be taken as 40 tons to the square inch of section for both tension and compression, then (using 4 as a factor for safety) what would be the maximum safe load uniformly distributed over such a girder, supposing it to be supported at each end, the supports being 12 feet apart ? Also make a diagram showing the distribution of the shearing stress in the middle transverse section.

14. A rectangular beam of timber is supported at both ends, and loaded by a weight in the centre. Make the necessary calculations for measuring the strength of the beam to resist breaking   For a bar of larch 6 feet long by 2 inches square, supported as above, the breaking weight is 515 lbs. ; taking this datum, you are required to solve the following question :—A cistern containing 2 tons of water rests on two cantilevers of larch, each 4 feet long and 5 inches in depth ; find the breadth of each cantilever. *Ans.* 1·85 inches.

15. A cast-iron beam of rectangular section, and having its lowest side horizontal, is supported at both ends. What difference should you make in the upper outline according as the load is evenly distributed or collected in the centre ?

16. A beam will safely carry a stationary load of 5 tons with a deflection of 2 inches, from what height may a weight of 200 lbs. be let drop upon the same beam without deflecting it to a greater extent ? *Ans.* 54 inches.

17. A steady load of 10 tons, suspended at the centre of a beam, deflects it through ⅜ inch. From what height would a weight of 300 lbs. require to fall in order to produce a like deflection when dropping on the beam ? *Ans.* 22·7 inches.

18. A cylindrical iron beam is 15 inches in its external diameter, and the metal is 1½ inches in thickness. The beam is fixed at the two ends, and is 35 feet between the supports ; find the greatest load uniformly distributed that the beam will bear, the greatest safe stress on the metal being 9,000 lbs. per square inch. *Ans.* 1,306 lb. per ft. run.

19. Compare the resistance to bending of a wrought-iron I section beam

when the beam is placed like this **I**, and like this **━**. The flanges of the
beam are each 6 inches wide and 1 inch thick, and the web is ⅜ inch thick
and measures 8 inches between the flanges. *Ans.* 4·56 : 1.

20. A horizontal bar of round iron, 1 inch diameter, 6 feet long, hinged
at the ends, is subjected to equal and opposite pushing forces of 1,000 lbs.
at its ends, and a load of 10 lbs. is hung at the middle so that it is both a
beam and a strut. Find the greatest stress anywhere. $E = 29 \times 10^6$ lbs.
per square inch. *Ans.* B.M. = 284·6 in.-lb. ; max. stress = 4,170 lbs. per
square inch.

21. Draw the bending-moment diagrams, and state the maximum bend-
ing moments for the six standard cases of loading and supporting a beam
of the same length, same load. (1) Fixed at one end, loaded at the other.
(2) Fixed at one end, loaded uniformly. (3) Supported at the ends, loaded
in the middle. (4) Supported at the ends, loaded uniformly. (5) Fixed
at the ends, loaded in the middle. (6) Fixed at the ends, loaded uniformly.

22. A uniform beam is fixed at its ends, which are 20 ft. apart. A load
of 5 tons in the middle ; loads of 2 tons each at 5 ft. from the ends. Find
the diagram of bending moment and prove your rule. State what the
maximum bending moment is, and where are the points of inflexion. *Ans.*
B.M. = 17·5 ton-feet. Points of inflection 4 feet from each end.

23. A rectangular beam, loaded in the middle, supported at the ends ;
find the shear stress at any point in any section. Find the deflection at
the middle, and distinguish between the parts due to ordinary bending and
to shear.

24. What occurs at the cross-section of a horizontal beam, carrying
vertical loads ? Where is the neutral line ? What is the value of the stress
at any place ? What is meant by *bending moment ?* Describe any model
which illustrates, however roughly, what occurs at a section of the beam.

25. A symmetrically loaded beam of uniform section ; given the diagram
of bending moment when supported at the ends, what is the easy rule for
finding the diagram when the beam is fixed at the ends ? Prove the rule
to be correct.

26. A beam of timber 2 feet long, 3 inches square, supported at the
ends and loaded at the middle, breaks with a load of 7,500 lbs. What
load may be expected to break a beam of the same material, fixed at one
end and loaded at the other, length 10 feet, breadth 5 inches, depth
9 inches. If the specimen beam deflected 0·034 inch for a load of
1,000 lbs., what would be the deflection of the second beam for a load
of 200 lbs. ? *Ans.* W = 5,625 lbs. ; δ = ·45 inch.

27. Suppose the vertical loads and supporting forces of a horizontal
beam to be known, show how we find (1) the shearing force at a section,
(2) the position of the neutral line, (3) the compressive stress at any part
of the section, and (4) the curvature of the beam. Prove your statements.

28. What are the functions of the top and bottom booms and of the
diagonal pieces of a railway girder ? Why are the booms usually larger in
section towards the middle of the girder, and the diagonal pieces larger
towards the ends of the girder ?

29. A rolled joist 12 inches deep, with flanges 4 inches wide and 1 inch
thick, and a web ½ an inch thick, carries on a 14-foot span a distributed
load of 1 ton per foot run, and a single concentrated load in the centre of
2¼ tons. Determine (*a*) the maximum bending moment and shearing
force ; (*b*) the maximum tensile and compressive stresses per square inch
at the centre section. *Ans.* Max. B.M. = 32·4 ton-ft. ; max. shear = 8·12
tons. Max. stress 44,400 lbs. per square inch. (This would exceed the
elastic limit, so joist is unsafe.)

LECTURE III.—I.C.E. QUESTIONS.

1. A girder is built of an ⊥ rolled joist; flanges 1 inch thick, 7½ inches wide ; web ⅝ inch thick, 20 inches deep ; with two ⅜-inch plates 12 inches wide riveted to each flange.   Calculate the modulus of the section (a) approximately, (b) more accurately.

2. Prove an expression for the deflection at mid-span of a beam resting freely on two supports and loaded uniformly per foot run ; or describe carefully how you would find it graphically.

3. A bridge is carried by two mild steel plate-girders over a span of 50 feet, the depth of the girder being 5 feet.   It is to carry an equivalent distributed load of 3 tons per foot run.   Selecting suitable data, design the section of the girder at the centre and at the ends.

4. A load W is applied in the middle of a cantilever of uniform section. Prove, either graphically or mathematically, that the elastic deflection of the free end is $\dfrac{5}{48}\dfrac{W\,l^3}{IE}$, l being the length of the cantilever, I the second moment of its section, and E the Young's modulus of the material.

5. State clearly the assumptions on which are based the usual formulæ for beams $\dfrac{M}{I}=\dfrac{f}{y}=E\dfrac{d^2y}{dx^2}$, noting to what extent and within what limits they are true.   A cast-iron beam, 2 inches × 1 inch, when subjected to a transverse test, broke when the tensile stress (calculated from the formula $M = fZ$) was 35,000 lbs. per square inch.   A similar specimen tested in tension broke when the stress was 28,000 lbs. per square inch.   Why do the results differ ?

6. The pitch or flute of a trough decking ($\_/\,\overline{}\,\diagdown\_$) is 32 inches and the depth is 13 inches, the neutral axis being at the half depth.   The moment of inertia round the neutral axis is 883 inch-units.   Find the distributed load per square foot which can be supported on a span of 30 feet if the stress is not to exceed 5 tons per square inch.

7. Describe briefly the reasoning and the hypothetical assumptions on which the ordinary theory of transverse flexure is based, and its deductions in regard to transverse strength—illustrating by the case of a simple beam of rectangular section.

8. If any beam of uniform section deflects 1 inch in a span of 100 inches under a central load, what will be the slope of the beam at each end ?

9. A steel girder crossing a span of 120 feet carries a uniform load, and is designed with a uniform depth of 10 feet, but with flanges of varying section adapted to a constant working stress of 6½ tons per square inch.   Calculate the deflection, taking E = 13,000 tons per square inch.

10. If the girder in the previous question were designed for the same span and load, and with the same depth, but were made of uniform section with a maximum flange stress of 6½ tons per square inch, what would be its deflection ?

11. A rectangular beam is loaded by a weight at its centre.   How will the deflection be changed by (a) doubling the span, (b) doubling the width, (c) doubling the depth, (d) doubling the load ?

12. If a bar of metal 2 inches square be placed upon bearings 3 feet apart, will it carry the greatest load at the centre with the sides vertical or with the diagonal vertical ?   Give the breaking-load for both positions when the bar is made of cast iron.

13. State the rule for finding the moment of inertia of a beam section which admits of being split up into a number of rectangular areas, and apply it to the case of a 4-inch × 4-inch tee-iron whose mean thickness is ½ an inch.

14. A mild-steel joist, total depth 6 inches, flanges 4 inches by ⅜ inch, and thickness of web 4/16 inch, rusts away to a depth of 1/16 of an inch all over the surface. Before rusting the load produced a stress of 8 tons per square inch on the outer layer. What will be the stress after rusting if the load is the same as before? The results need only be given approximately.

15. Describe with sketches the method you would employ to determine the deflection produced by a passing train in the main girder of a plate girder bridge of about 60 feet span over a road.

16. Either of the following sections is available for a beam which is required to be as strong and as stiff as possible :—(a) Circular 2 inches in diameter, or (b) rectangular 2 inches deep, 1·178 inches wide. Which would you use?
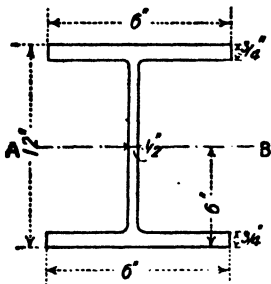
17. The deflection of two plate girders is to be ascertained when a load of 10 tons acts at the centre. An hydraulic jack capable of exerting a push of 5 tons is available ; it is fitted with a pressure gauge, so that the pressure it exerts can be measured. Sketch the arrangement you would employ to determine the deflection.

18. A cast-iron water pipe 10 inches external diameter and ½ inch thick rests on supports of 40 feet apart. Calculate the maximum stress in the outer fibre of the material when empty and when full of water, also the corresponding deflections.

19. Calculate the moment of inertia round the axis A B of the rolled joist of the section shown in the sketch, assuming that the flanges and web are rectangular.

LECTURE IV.

Torsional Strength of Shafts.— In order to transmit energy
through a shaft, the driving force must be applied at some
distance from its centre. The driving force and its effective
leverage, therefore, constitute what is termed a **Turning or
Twisting Moment** (T.M.) which puts the shaft in a state of
twist or torsion. The tendency of a purely torsional moment
applied to a shaft is to cause the shaft to shear in planes
normal to its axis, and this has to be met by the shearing
resistance of the material, which resistance must, of course,
be of the nature of a moment. The resistance the shaft offers
to twisting we term its **Torsional Resistance** (T.R.); and as this
balances the turning moment, we have :—

$$\text{T.M.} = \text{T.R.}$$

We have now to find the value of T.R., as depending on the
material and dimensions of the shaft, and shall confine ourselves
to shafts of circular section—solid and hollow. Suppose the
accompanying figure to represent an end view of a shaft ; and
suppose A B and $a\,b$ to have been parallel diameters of two
sections very near to each other when the shaft was at rest ;
then, when the shaft is at work trans-
mitting energy, the diameters, A B and
$a\,b$, will no longer be parallel, but will
make an angle with each other, as
shown. A longitudinal section, through
the axis of a shaft, which is a plane
when the shaft is at rest, thus becomes
a screw surface when the shaft is
working. We shall have occasion later
to measure this angle of twist ; but in
the meantime we are mainly concerned
with the distribution of shearing stress
within the shaft.
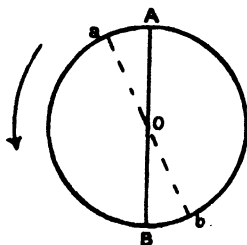
ILLUSTRATING STRAIN
IN A SHAFT.

Looking at the figure, we easily see that the *strain* in any
ring of fibres must be proportional to the arc of this ring

which is included between the diameters A B and $a\,b$, when these are twisted out of parallelism by the turning moment. Within the elastic limit of the material, therefore, it follows that the shearing *stress* in any ring of fibres is proportional to the radius of that ring.

Therefore, let f = the greatest shearing stress, in lbs. per sq. inch, permissible in the material of the shaft.

D = the outside diameter,

d = the inside diameter of the shaft, both in inches.

And x = the radius of *any* ring of fibres within the material of the shaft.

Then the shaft must be so proportioned that f shall be the value of the stress in its outermost fibres which are $\frac{1}{2}$ D inches from the centre. Consequently, from what has already been said, we have :—

$$\text{Stress at } x = \frac{x}{\frac{1}{2}\,D}\,f = \frac{2\,x}{D}\,f.$$

Consider, now, the ring of fibres at x inches from the shaft centre, whose radial thickness is $d\,x$ inches. The sectional area of this elementary ring will = $2\,\pi\,x\,d\,x$ sq. inches; and its resistance to shearing will be

$$2\,\pi\,x\,d\,x \times \frac{2\,x}{D}\,f \text{ lbs. } = \frac{4\,\pi f}{D}\,x^2\,d\,x \text{ lbs.}$$

Now, the leverage at which this resisting ring of fibres acts, is x inches; therefore, its *moment* of resistance is $\frac{4\,\pi f}{D}\,x^2\,d\,x \times x$,

or $\frac{4\,\pi f}{D}\,x^3\,d\,x$ inch-lbs.

Hence, summing up the moments of resistance of all such elementary rings which go to make up the shaft, we get:—

$$\text{T.R.} = \frac{4\,\pi f}{D} \int_{\frac{1}{2}d}^{\frac{1}{2}D} x^3\,d\,x,$$

$$" \quad = \frac{4\,\pi f}{D} \left\{ \frac{(\frac{1}{2}\,D)^4}{4} - \frac{(\frac{1}{2}\,d)^4}{4} \right\}$$

$$" \quad = \frac{\pi}{16}\,f \left(\frac{D^4 - d^4}{D} \right).$$

Hence, for hollow shafts,* we have :—

$$\text{T.R.} = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)f \quad . \quad . \quad . \quad . \quad \text{(I)}$$

For solid shafts, we make $d = 0$, and get :—

$$\text{T.R.} = \frac{\pi}{16}D^3 f \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

It is instructive to compare the torsional resistances of solid and hollow shafts of the same weight and material. For this purpose let D_1 be the outer diameter of hollow shaft.

Then, if we neglect couplings, and consider the shafts to be of equal length, the weights will simply be proportional to their sectional areas ; i.e.:—

$$\frac{\text{Weight of hollow shaft}}{\text{Weight of solid shaft}} = \frac{D_1^2 - d^2}{D^2}.$$

For equal weights, this ratio is unity ; therefore we have the relation :—

$$D^2 = D_1^2 - d^2$$

Or, $$D = \sqrt{D_1^2 - d^2}.$$

Now, we have from equations (I) and (II) :—

$$\frac{\text{T.R. of hollow shaft}}{\text{T.R. of solid shaft}} = \frac{D_1^4 - d^4}{D_1 \times D^3} = \frac{D_1^2 + d^2}{D_1 \times D} \times \frac{D_1^2 - d^2}{D^2}$$

$$\text{''} \qquad \text{''} \qquad = \frac{D_1^2 + d^2}{D_1 \times D} = \frac{D_1^2 + d^2}{D_1 \times \sqrt{D_1^2 - d^2}}.$$

It will simplify matters if we put $d = x \times D_1$, where x is a proper fraction, we then have :—

$$\frac{\text{T.R. of hollow shaft}}{\text{T.R. of solid shaft}} = \frac{1 + x^2}{\sqrt{1 - x^2}}.$$

For example, let $x = \frac{1}{2}$, then :—

$$\frac{1 + x^2}{\sqrt{1 - x^2}} = \frac{1 + \frac{1}{4}}{\sqrt{1 - \frac{1}{4}}} = \frac{5}{2\sqrt{3}} = 1.443.$$

This result shows that for the same length and weight, the hollow shaft having outer and inner diameters in the proportion of 2 to 1 will be 44.3 per cent. stronger than the solid one.

The turning moment driving a shaft may either be uniform or variable in amount. Shafts driven by means of gearing, and revolving at a uniform speed, are generally considered as cases

* See *Journal Junior Inst. of Engs.*, vol. xviii., 1907-8, for Prof. Lilly's paper on " The Economic Design of Hollow Shafts."

of uniform turning moment. As a typical example of variable turning moment, we have the case of the steam engine crankshaft, where both the driving force of the steam on the piston and its effective leverage are continually varying throughout the stroke.

When the turning moment is uniform—that is, when the shaft revolves uniformly at n revolutions per minute, and transmits energy at the rate of so many H.P., this is all the data we require to know in order to estimate T. M. We have already seen (see Vol. I., Lect. III.) that the work done by a turning couple in one minute is equal to the magnitude of the turning couple multiplied by its angular displacement in the same time. Now our turning couple, or turning moment, as we call it, is T.M. inch-lbs., or $\frac{1}{12}$ T.M. foot-lbs., and the angular velocity of our shaft is $n \times 2\pi$ radians per minute.

Therefore, the

$$\text{Work done} = \frac{\text{T.M.}}{12} \times 2\pi n \text{ ft.-lbs. per minute}$$

And the

$$\text{H.P.} = \frac{\frac{\text{T.M.}}{12} \times 2\pi n}{33,000} = \frac{n \times \text{T.M.}}{63,025}.$$

∴

$$\text{T.M.} = 63,025 \cdot \frac{\text{H.P.}}{n}. \qquad \ldots \ldots \text{(III)}$$

EXAMPLE I.—Find the moment of resistance to torsion of a hollow shaft. Compare the strengths to resist torsion of a solid and hollow shaft of the same length and weight, the extreme diameter of the hollow shaft being double its internal diameter. A hollow shaft, the external and internal diameters of which are 20 inches and 8 inches respectively, runs at 70 revolutions per minute, with a surface stress of 6,000 lbs. per square inch; find the twisting moment and the horse-power transmitted.

ANSWER.—The first two parts of this question have already been answered in the text.

With regard to the last part, we are asked to find the values of T.M. and H.P., being given :—

$$D_1 = 20 \text{ inches.} \qquad f = 6,000 \text{ lbs. per sq. in.}$$
$$d = 8 \text{ inches.} \qquad n = 70 \text{ revs. per min.}$$

Since

$$\text{T.R.} = \text{T.M.},$$

$$" = \frac{\pi}{16} \cdot \frac{D_1^4 - d^4}{D_1} \cdot f$$

7

$$\text{T.M.} = \frac{3\cdot1416}{16} \times \frac{20^4 - 8^4}{20} \times 6000.$$

$$\text{,,} = 9,184,000 \text{ inch-lbs.,}$$

and

$$\text{H.P.} = \frac{\text{T.M.} \times n}{63,025},$$

$$\text{,,} = \frac{9,184,000 \times 70}{63,025}$$

$$\text{,,} = 10,200.$$

EXAMPLE II.—If a steel shaft revolving at 60 revolutions per minute be required to transmit 220 horse-power, what should be its diameter so that the maximum stress produced in it may not exceed one-fifth of that at the elastic limit? The elastic limit in torsion is 18 tons per sq. inch. Prove any formula you may employ.

ANSWER.—Combining formulæ (II) and (III) we have :—

$$\text{T.R.} = \text{T.M.},$$

i.e.,

$$\frac{\pi}{16} D^3 f = 63,025 \times \frac{\text{H.P.}}{n}.$$

$$D = 68\cdot5 \ \sqrt[3]{\frac{\text{H.P.}}{n f}}. \quad \cdots \quad \text{(IV)}$$

Here, H.P. = 220. n = 60.

And, $\quad f = \frac{1}{5} \times 18 \times 2240 = 8064$ lbs. per sq. in.

$$D = 68\cdot5 \times \sqrt[3]{\frac{220}{60 \times 8064}} = 5\cdot27 \text{ inches.}$$

In cases where the turning moment exerted on a shaft varies, it is, of course, necessary that the shaft should be of strength sufficient to withstand safely the maximum value of T.M. So that in dealing with an example like that of the steam engine crank-shaft we take as the turning force the product of the maximum effective steam pressure on the piston into the piston area; and for the leverage we take the crank radius, although this is not quite accurate ; because, if the crank be driven by means of a connecting-rod, the virtual leverage of the steam force at a certain point in the stroke exceeds that of the crank radius by an amount depending on the relative lengths of the crank and connecting-rod.

EXAMPLE III.—SHAFTS. 99

But on the other hand, the effective steam pressure on the piston is, as a rule, much below its maximum value when the piston reaches the point of greatest leverage. On the whole, therefore, it is quite accurate enough for all practical purposes to estimate the maximum turning moment in the way we have indicated.

Thus, Let p = Greatest effective steam pressure acting on the piston, in lbs. per sq. inch.

„ A = Area of piston, in sq. inches.

„ r = Crank-radius, in inches.

Then, max. T.M. = $p\,A\,r$ inch-lbs.

By *effective* steam pressure, we mean the *difference* between the pressures behind, and in front of, the piston.

EXAMPLE III.—Find the diameter of the crank-shaft for a horizontal engine which is to be worked with an effective mean steam pressure of 45 lbs. per square inch throughout the stroke, the diameter of the cylinder being 36 inches, the stroke 5 feet, and the working load being taken at $\frac{1}{8}$ of the breaking load. The shaft is to be of wrought iron, such that a 1-inch shaft will break with the torsion produced by 800 lbs. acting at the end of a 12-inch lever.

ANSWER.—Let f_{B} be the breaking stress of the experimental shaft, then the working stress in the crank shaft, according to the question, will be $\frac{1}{8} f_{B}$.

To find the value of f_{B} we are given that when T.M. = 800 × 12 inch-lbs., and D = 1″, fracture takes place. From these data, therefore, we deduce :—

$$f_{B} = \frac{800 \times 12}{\dfrac{\pi}{16} \times 1^{3}} = \frac{800 \times 12}{\dfrac{\pi}{16}} \text{ lbs.}$$

The area of a 36-inch piston = 1017·87 square inches, and r is 30 inches.

$\therefore$ Max. T.M. = 45 × 1017·87 × 30 inch-lbs.

Also, „ = $\frac{\pi}{16} D^{3} f$.

$\therefore$ $D^{3} = \dfrac{45 \times 1017\cdot87 \times 30}{\dfrac{\pi}{16} f}$;

but, $f = \frac{1}{6} f_{k} = \dfrac{800 \cdot \times 2}{\dfrac{\pi}{16}}.$

Hence, $D = \sqrt[3]{\dfrac{45 \times 1017 \cdot 87 \times 30}{800 \times 2}},$

                „ $= 9 \cdot 5$ inches, nearly.

Strength of Shafts subjected to combined Twisting and Bending.—In Example III. the diameter of the shaft has been calculated as for a purely twisting moment. But in no case of a shaft being driven by a crank is the effect of the load quite so simple as this. Besides the turning moment, which we have already seen how to deal with, there is always in action a *bending* moment of greater or less magnitude depending on the engine arrangement. The worst case is that in which the crank is overhung. When this is so, the bending moment is caused by the load on the piston acting along a line (the centre line of the cylinder) at a certain distance from the shaft bearing nearest to the crank.

Let $l =$ the distance between the centre line of the cylinder and the middle of the nearest shaft bearing, in inches ; and

p and $A =$ (as before) the effective steam pressure and piston area respectively.

Then the magnitude of the bending moment which we have now to take into account is

$$B.M. = p \, A \, l \text{ inch-lbs.}$$

This bending moment is balanced by the moment of resistance of the shaft, which, as will be shown in Lecture III., is—

$$R.M. = \frac{\pi}{32} D^{3} f_{t} \, ;$$

Where, $D =$ diameter of the shaft journal, in inches,

And, $f_{t} =$ the tensile stress in the outer fibres of the journal, in lbs. per sq. inch.

Hence, we see that when a crank-shaft is being turned by the steam on the piston, it is subjected simultaneously to a shearing stress of intensity f_{s}, and a tensile stress of intensity f_{t}. The problem now before us is to combine these stresses so as to

obtain what is termed the *Equivalent* tensile or shearing stress; but in order to render all the steps in the process clear and intelligible, we require to demonstrate the following theorem :—

Theorem.—A shearing stress on any plane produces a shearing stress of equal intensity on planes at right angles to it.

Let A B O D be a rectangular block of material whose thickness is 1 inch perpendicular to the plane of the paper. And let f_s be the intensity of the shearing stress over the face whose edge is
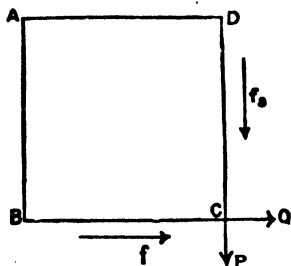
ILLUSTRATING SHEARING STRESS THEOREM.
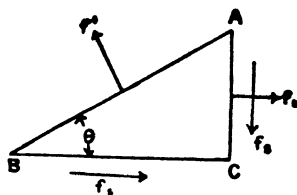
ILLUSTRATING EQUIVALENT TENSILE STRESS.

O D. It is easy to see that the total shearing force on the face O D which tends to pull that face parallel to itself, must be accompanied by a similar effect on the face B C in order that the block may not be turned around A. To find the relation between those forces, take moments about A, and we get :—

$$P \times A D = Q \times A B.$$

Or, $$(f_s \cdot C D) \times B C = (f \cdot B C) \times C D.$$

∴ $$f_s = f.$$

Hence, we see that the shearing stress induced in a shaft by the turning moment is accompanied by a shearing stress of equal intensity on planes at right angles to it ; that is, parallel to the axis of the shaft.

In the right-hand figure let A C represent the edge of a small portion of a plane normal to the axis of the shaft, and B C that of another plane at right angles to A C. On the former of these planes there is a shearing stress of intensity f_s, due to the turning moment, and a direct tensile stress of intensity f_t, due to the bending moment acting on the shaft. By the theorem just proved, we also have on B C a shearing stress f_s. Let f' denote the intensity of a tensile stress, which, acting on a

plane A B inclined to A C and B C, would balance the stresses
on these latter planes. As before, let the width of the three
planes perpendicular to the plane of the paper be unity.

Resolving vertically and horizontally, we have :—

$$(f' \cdot A B) \cos \theta = (f_s \cdot A C),$$

and $$(f' \cdot A B) \sin \theta = (f_s \cdot B C) + (f_t \cdot A C).$$

From the first of these equations we get :—

$$\frac{f'}{f_s} = \frac{\dfrac{A C}{A B}}{\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta, \quad \ldots \ldots \quad (1)$$

and from the second :—

$$f' = \frac{\dfrac{B C}{A B}}{\sin \theta} \cdot f_s + \frac{\dfrac{A C}{A B}}{\sin \theta} \cdot f$$

$$,, = \frac{\cos \theta}{\sin \theta} f_s + \frac{\sin \theta}{\sin \theta} f_t$$

$$,, = f_s \cot \theta + f_t$$

Or, $$\frac{f' - f_t}{f_s} = \cot \theta. \quad \ldots \ldots \ldots \quad (2)$$

Multiplying together (1) and (2), we get:—

$$\frac{f'}{f_s} \cdot \frac{f' - f_t}{f_s} = 1.$$

∴ $$f' (f' - f_t) = f_s^2.$$

Which on being solved for f' gives :—

$$f' = \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2} \quad \ldots \ldots \ldots \quad (V)$$

We take the positive sign in the solution of this quadratic
equation, for obviously f' is greater than $\frac{1}{2} f_t$.

Being now in possession of the relation subsisting among the
stresses, we next have to express these in terms of the T.M.
and B.M. :—

Since $\text{T.M.} = \dfrac{\pi}{16}\ D^3\, f_s$

And $\text{B.M.} = \text{R.M.} = \dfrac{\pi}{32}\ D^3 f_t$

Hence, $f_s = \dfrac{\text{T.M.}}{\dfrac{\pi}{16}\ D^3}$ . . . . . (3)

And $f_t = \dfrac{\text{B.M.}}{\dfrac{\pi}{32}\ D^3}$ . . . . . (4)

In like manner, we must have :—

$$f' = \dfrac{\text{B.M.}}{\dfrac{\pi}{32}\ D^3}$$

where B.M.' stands for the *equivalent* bending moment.
   Making these substitutions, and reducing, (V) becomes :—

$$\text{B.M.}' = \tfrac{1}{2}\ \{\text{B.M.} + \sqrt{\text{B.M.}^2 + \text{T.M.}^2}\}\quad . \quad . \quad . \ (VI)$$

   If we follow Guest's theory (p. 107 onwards), we should find
the equivalent twisting moment by the formula—

$$\text{T.M.}' = \sqrt{\text{B.M.}^2 + \text{T.M.}^2}. \quad . \quad . \quad . \ (VII)$$

EXAMPLE IV.— Investigate an expression in terms of f_t,
f', and f_s, which will give the resultant tensile stress, f',
per square inch of section in a material which is subjected
at the same time to a direct tensile stress of f_t lbs. per
square inch, and to a shearing stress, f_s lbs., per square
inch.   A bar of iron is at the same time under a direct
tensile stress of 5,000 lbs. per square inch, and to a shearing
stress of 3,500 lbs. per square inch.   What would be the
resultant equivalent tensile stress in the material?
   ANSWER.—The complete investigation referred to in the first
part of this question is given in the text, and equation (V)
is the expression required.   It only remains to find the
numerical value of f', having given

$$f_t = 5{,}000 \text{ and } f_s = 3{,}500,$$

$$\therefore \qquad f'' = \frac{5,000}{2} + \sqrt{\frac{5,000^2}{4} + 3,500^2},$$

$$,, = 6,800 \text{ lbs. per square inch fully.}$$

EXAMPLE V.—A wrought-iron shaft is subjected simultaneously to a bending moment of 8,000 inch-lbs., and to a twisting moment of 15,000 inch-lbs. Find the twisting moment equivalent to these two, and the least safe diameter of the shaft. The safe tensile stress is to be taken at 8,000 lbs. per square inch. Prove clearly the formula you employ.

ANSWER.—Here we have :—

$$\text{B.M.} = 8,000 \text{ inch lbs.,}$$

and $\qquad$ T.M. = 15,000 $\qquad$,,

Hence, by formula (VII) we get :—

$$\text{B.M.}' = \tfrac{1}{2} \, (8,000 + \sqrt{8,000^2 + 15,000^2}),$$

$$,, = 12,500 \text{ inch-lbs.}$$

To find the diameter of the shaft to withstand this B.M.' with a tensile stress of not over 8,000 lbs. per square inch, we employ formula (II) making :—

$$\text{B.M.} = \frac{\pi}{32} \, \text{D}^3 f.$$

$$\therefore \qquad \text{D} = \sqrt[3]{\frac{\text{B.M.}}{\frac{\pi}{32} \cdot f}} = \sqrt[3]{\frac{12,500}{\frac{3 \cdot 1456}{32} \times 8,000}}$$

$$= 2 \cdot 51 \text{ inches.}$$

Stiffness of Shafts.—Angle of Twist.—We have already seen that the effect of a turning moment applied to a shaft is to twist one part relatively to another. Hitherto we have been dealing only with the resistance the shaft offers to being twisted—that is to say, we have been concerned only with the *strength* of the shaft without regard to the question of *stiffness*. In many cases

—especially in light machinery—the question of the stiffness of the shafting is of greater importance than that of strength.

The stiffness of a shaft is measured by the smallness of the angle of twist per unit length of the shaft.

Turning back to the figure illustrating strain in a shaft, let dl be the axial distance, in inches, between the two sections whose diameters are A B, ab, and let $d\theta$ be the circular measure of the angle between those diameters when the shaft is twisted; then the torsional, or shearing *strain* at the surface of the shaft, is

$$= \left(\frac{D}{2}\right) \times \frac{d\theta}{dl}.$$

D, as before, being the extreme diameter of the shaft in inches,

Let f = Surface stress in the material of the shaft in lbs. per sq. inch.

„ C = Modulus or coefficient of shearing elasticity or of rigidity in lbs. per sq. inch.

Then, since

$$C = \frac{\text{stress}}{\text{strain}} = \frac{f}{\left(\dfrac{D}{2}\right) \cdot \dfrac{d\theta}{dl}}.$$

∴ $$d\theta = \frac{2f}{CD} \cdot dl.$$

Hence, for a shaft L inches long we have, by a simple integration, the angle of twist.

$$\theta = \frac{2f}{CD}\int_0^L dl = \frac{2fL}{CD}.$$

To express this result in terms of the twisting moment and the diameter of the shaft, we have :—

$$f = \frac{\text{T.M.}}{\dfrac{\pi}{16}D^3} \quad \text{for solid shafts.}$$

And, $$f = \frac{\text{T.M.}}{\dfrac{\pi}{16}\dfrac{D^4 - d^4}{D}} \quad \text{for hollow shafts}$$

Making these substitutions and simplifying, we get :—

Angle of twist for solid shafts,

$$\theta = \frac{10 \cdot 2 \,(\text{T.M.})\,L}{C\,D^4} . \text{ radians.}$$

Or,

$$\theta^\circ = \frac{584\,(\text{T.M.})\,L}{C\,D^4} \text{ degrees.}$$

(VIII)

And, for hollow shafts,

$$\theta = \frac{10 \cdot 2 \,(\text{T.M.})\,L}{C\,(D^4 - d^4)} . \text{ radians.}$$

Or,

$$\theta^\circ = \frac{584 \,(\text{T.M.})\,L}{C\,(D^4 - d^4)} . \text{ degrees.}$$

(IX)

By the equations just established, we see that, while the strength of shafts vary as the *third* power of their diameters, their stiffness varies as the *fourth* power.

EXAMPLE VI.—Establish a formula for the moment of resistance to torsion of a solid shaft of circular section. The angle of torsion of a shaft is limited to 1° for each 10 feet of length ; find the diameter of a solid round shaft to transmit 100 H.P. at 50 revolutions per minute, the modulus of resistance to torsion being 10,000,000 lbs. per sq. inch.

ANSWER :—

Here, $\theta = 1°$ when, $L = 10 \times 12 = 120$ inches

And, $C = 10,000,000.$

Also, $\text{T.M.} = 63,024 \times \dfrac{\text{H.P.}}{n} = 63,024 \times \dfrac{100}{50}$

„ $= 126,048$ inch-lbs.

Now, applying formula (VIII) the given conditions are that :—

$$1° = \frac{584 \times 126,048 \times 120}{10,000,000 \times D^4}$$

Hence, solving for D, we get :—

$$D = \sqrt[4]{\frac{584 \times 126,048 \times 120}{10,000,000}} = 5 \cdot 45 \text{ inches.}$$

The Strength of Ductile Materials under Combined Stress.—
One of the most original, thorough, and important papers
dealing with the Strength of Materials under the above
heading was read by Mr. James J. Guest, M. I. Mech. E., &c.,
Birmingham, before the Physical Society of London on May
25th, 1900.*

Mr. Guest personally carried out a large number of experiments,
chiefly upon metal tubes, by subjecting them to torque, to torque
and tension combined, to tension only, to tension and internal
fluid pressure, to torsion and internal fluid pressure, and to
internal fluid pressure only. He took most careful observations
of the axial elongation, of the twist, and of the circumferential
strain, &c., by means of apparatus specially designed by himself,
which are all illustrated and described in his paper. He
also gives sufficient mathematics to prove the action and
accuracy of his measuring apparatus, &c., eight tables of results,
and a number of curves plotted from these results.

Here we must be content with quoting his first and his last
articles. In the latter he gives what may be fairly called
"**Guest's Law,**" which we have therefore printed in italic type,
because the truth of his practical conclusions have been since
proved by other careful experimenters.

"1. *Lack of Knowledge of the Laws of Strength.*—From the
point of view of both pure and applied science, it would be of
interest to know the complete laws of the strength of materials;
but although a multitude of tests have been made in certain simple
modes, our knowledge of the laws of strength has been extended
by few experiments exposing the material to two or more prin-
cipal stresses, and thus, except in the simpler cases, the elastician
is without experimental guidance as to what he should seek
analytically as determining the strength of the body under
consideration.

" In the series of experiments herein described the materials
employed have been subjected to a certain variety of simul-
taneous principal stresses; and the results are recorded in
the hope that they will prove of service to elasticians and
engineers.

"The simplest, and most primitive, method of ascertaining the
strength of a material is to subject a cylindrical specimen of it
to a direct tension or compression, and to increase the force
until the specimen breaks or collapses; the breaking stress,
thus found, being taken as the basis for calculations of the
strength of all pieces of that material used in structures or
machines.

* See Note on next page.

"As material is frequently exposed to torsion, another frequently-employed test is to break a circular cylinder of material by the application of a torque, the stress so found being used in calculations for shafts, &c."

Note.—This paper was communicated by the Physical Society to the London, Edinburgh, and Dublin *Philosophical Magazine and Journal of Science.* It is printed in Vol. L.—Fifth Series, July to December, 1900. It occupies 63 pages, and is divided into the following 58 different articles :—

CONTENTS OF ARTICLES.

1. Lack of Knowledge of Laws of Strength.
2. Separation of the Isotropic Materials into Ductile and Brittle.
3. The Yield Point in Tension Tests.
4. The Yield-point, rather than the Ultimate, Stress the Criterion of Strength.
5. Variations of the Ultimate and Yield-point Stresses : Annealing.
6. The Elastic Limit.
7. The Elastic-limit Effect probably due to the Existence of Local Yield Points.
8. The Yield Point, in Preference to the Elastic Limit, selected as the Criterion of Strength.
9. Torsion a Case of Combined Stress.
10. First Reason for Use of Thin Tubes as Specimens.
11. Results of previous Torsion Experiments.
12. Previous Experiments upon the Yield Point under Combined Stress.
13. Experiments upon Ultimate Strength under Combined Stress.
14. Theories of Elastic Strength under Combined Stress : the Maximum - stress Theory.
15. The Maximum-strain Theory.
16. Theorem upon the Limiting Values of σ. (It represents Poisson's ratio.)
17. The Maximum-strain Theory not Disproved by Published Experiments.
18. The Maximum Shearing-stress Criterion of Elastic Strength.
19. Further Reasons for Adopting Thin Tubes as Specimens.
20. Range of Stress covered by Different Types of Experiment.
21. Method of Checking the Isotropism of the Material as regards Yield-point Stresses.
22. Objections to Tubes on Account of Want of Isotropy.
23. Objections to Tubes on Account of Large Effect of Defects.
24. The System of Tests.
25. The Specimens and Holders.
26. The Tension Loads.
27. The Application of Torque.
28. The Friction of the Torsion-rigging.
29. The Application of the Internal Pressure.
30. Measurement of the Fluid Pressure.
31. A New Type of Pressure Gauge.
32. Measurement of the Distorsions.
33. A New Extensometer.
34. Errors of Extensometer.
35. A New Twist-measuring Apparatus.
36. Proof that the Torsion-mirrors Measure the Twist.
37. Effect of Bending.
38. As a Transmission Dynamometer.
39. The Diametral Extensometer.
40. Method of Making the Tests.
41. The Determination of the Sectional Area and Thickness of the Tubes.
42. Calculation of the Stresses.
43. Calculation of the Strains.
44. Maximum Shear and Slide are Proportional.
45. Quantities Tabulated.
46. Results of the Experiments—The Tests on Solid Bars.
47. The Material and Shape of Tubes Satisfactory.
48. The Elastic Limit Phenomenon.
49. Yield-point Stresses of the same type nearly Constant throughout Series on each Tube.
50. The Variation of the Maximum Stress.
51. The Variation of the Maximum Principal Strain.
52. The Maximum Shearing Stress or Slide nearly Constant.
53. Graphical Presentation of Variations.
54. Conclusions Probable for General Type of Stress.
55. Effect of a Volumetric Stress (Steel).
56. Convenient View of General Type of Stress.
57. The Copper and Brass Tests and Diagrams.
58. Practical Conclusions and Note.

As far as I am aware, this paper has not appeared in any other magazine, paper, or journal. But, most certainly, all students specially interested in this subject of combined stresses should endeavour to get the loan of this Vol. L. and study the paper most carefully. It is too long and too detailed to make even a full abstract thereof in this book.

"58. *Practical Conclusions and Note.*—The result, then, of these experiments, as applicable to practice, is, *that the condition for initial yielding of a uniform ductile material is the existence of a specific shearing stress. And, that the intermediate principal stress is without effect.*[*]

"*Note.*—Cases of more than one principal stress are met with most frequently in boilers, hydraulic cylinders, and crank-shafts. In the former cases the results obtained above lead to the same dimensions as are obtained by the consideration of the greatest stress only, while the maximum-strain theory would lead to the adoption of a smaller thickness. In the case of crank-shafts, or 'combined bending and twisting,' both the greatest stress and greatest strain theories lead to small dimensions determined from the tensile stress, the correspondng well-known formulæ for the 'equivalent bending moment' being

$$\tfrac{1}{2} (M + \sqrt{M^2 + T^2}) \quad \text{and} \quad \tfrac{3}{8} M + \tfrac{5}{8} \sqrt{M^2 + T^2}$$

respectively, where M is the applied bending moment and T the applied twisting moment. The greatest shearing-stress theory leads to the value $\sqrt{M^2 + T^2}$ for the equivalent twisting moment, and, where the dimensions are thus determined, that formula should be used.

"It may also be noted that the specific shearing stress at the yield point is better determined by taking one-half of the tensional yield-point stress, than from the results of torsional experiments in which the sharpness of the yield point is masked, as explained in § 10 and shown in Fig. 15 of the paper."

Mild-Steel Tubes in Compression and under Combined Stress.—Mr. William Mason, A.M.I.Mech.E., of the University of Liverpool, read a paper before The Institution of Mechanical Engineers on 17th Dec., 1909, upon this subject.[†] It begins as follows : —

"*Preliminary.*—One of the most important contributions during recent years to our knowledge of the strength of materials is the paper by Mr. J. J. Guest on 'The Strength of Ductile

---

[*] This part in italics may be fairly termed "*Guest's Law of Combined Stress.*"

[†] I am indebted to the Council of The Institution of Mechanical Engineers, not only for their kind permission to make the following abstracts from Messrs. Mason's and Smith's papers, but also for the copper block from which the next figure is printed. These are in the *Proceedings* of the above Institution at pp. 1205, &c., Parts 3-4, 1909. Also see *The Engineer*, Dec. 24 and 31, 1909, and *Engineering*, Dec. 24, 1909

Materials under Combined Stress.' The experimental results
and the conclusions deduced have gradually been brought to the
notice of engineers, and it is only necessary to re-quote Mr.
Guest's practical conclusion, namely, 'That the condition for
initial yielding of a uniform ductile material is the existence of
a specific shearing stress.' This result, if true, must be of great
practical importance; for in all cases of stress (with the excep-
tions of positive and negative fluid stress) there is shear stress
induced over certain planes.

"Other investigators* have made experiments in combined
stress which are confirmatory of Mr. Guest's conclusion. So far
as the author is aware, very little work has hitherto been done
in which the compressive stress has been much greater in
intensity than the other stresses. In Mr. Guest's tests a prin-
cipal stress, when compressive, never exceeded the other
principal stress in intensity. Mr. Scoble did work on bars
under bending and torsion, in which the yield may have been
due to the stress on either the compressed or extended side of
the bent bar; and Professor Hancock's reports only contained
seven tests on simultaneous compression and torsion.

"*Object and Scope of the Experiments.*—The work described in
this paper is largely upon comparative yield-point strengths in
compression and shear; though in some cases the stresses have
been carried as far as the rupture or collapse of the material.
In making tests in compression, it is essential that the force
shall be axial to the specimens. It appeared to the author that
this condition might be attained, approximately at any rate, by
the apparatus hereafter described, using tubes as specimens; and
that, as in Mr. Guest's experiments, a second stress, at right
angles to the direction of the compression, could be conveniently
imposed by hydraulic pressure in the interior of the tube.
Besides the sets of tests in simple compression, and in simultaneous
compression and hoop tension, three series of tests were made on
certain of the tubes under conditions as follows—namely, tests
under simple axial tension, simultaneous axial and hoop tension,
simple axial compression, simultaneous axial compression and
hoop tension, and simultaneous axial and hoop compression.

"*Material for Tests.*—The specimens were cut from solid-
drawn mild-steel tubes. Two sizes of tubes were used—namely,
3 inches bore, 14 gauge (about 0·08 inch thick), and 2¾ inches
bore, 10 gauge (about 0·128 inch thick); the former being cold-

* Professor E. L. Hancock, *Philosophical Magazine*, Feb. and Oct. 1906.
Mr. W. A. Scoble, *Philosophical Magazine*, Dec. 1906, and *Proceedings of
the Physical Society*, Sept. 1907. Mr. L. B. Turner, *Engineering*, 12th
Feb. 1909.

drawn and the latter hot-drawn. Various lengths of specimens of each of these were tested, the lack of uniformity noticeable in the lengths being due to the circumstance that it was difficult to cut, from the same piece, the required lengths of exactly straight tube. Most of the tubes were supplied by the British Mannesmann Tube Company. They had undergone some degree of annealing, but not so far that they could be considered isotropic for the purpose of the tests. Experiments were made on the tubes both in the condition as supplied, and also after further annealing. The latter was done by encasing the tubes in a longer tube of rather larger diameter, the ends of which were blocked up with asbestos and fireclay. The tube and the outer casing were kept from contact by narrow rings of asbestos. The whole was placed in a coke furnace at a temperature of from 800° to 850° C., and allowed to remain there about twenty-five minutes, the casing (and tube) being turned round about its axis from time to time.

" It was impossible to obtain tubes of exactly uniform thickness of wall. In a long length of tube the thickness would be practically uniform at one section, while at another section there might be a considerable variation. The maximum variation in thickness of wall of the tubes used in the tests was 0·007 inch, means being taken (see description of apparatus for compression tests) to obtain uniformity of compressive stress. The ends of all the tubes tested were accurately faced up in the lathe."

Mr. Mason then illustrates the apparatus and describes the method which he adopted of holding the tube ends during the application of tension, compression, shear, simultaneous axial and hoop compression, simultaneous compression and hoop tension, and the measurement of strains by Ewing's extensometers.*

The tests were made by the 100-ton Buckton testing machine in the Walker Engineering Laboratory of the University of Liverpool, under the supervision and encouragement of my old student and assistant, Professor Walkinson, M.Inst.C.E., &c. He explains the results of his tests, gives tables of the results, and diagrams showing the strengths of tubes when not annealed and after being annealed.

His *conclusion* is—" That the experiments show an approximate agreement between the maximum shear stress at the yield point in compression and the yield-point stress in pure shear, the mean difference in the tests of annealed specimens being about 3 per cent. It appears, then, that mild steel in direct compression yields by shearing; and, to a first approximation,

* See Index for the page in this book where this instrument is described.

that the value of this shear stress is independent of any normal compressive stress on the planes of the slide."

Compound Stress Experiments.—At the same meeting of the Institution of Mechanical Engineers, Prof. C. A. M. Smith, B.Sc., read a paper on this subject.[*]

" (I.) *Theories of Elastic Failure.*—The object of this paper is to give certain experimental results which have been obtained upon solid steel bars subjected to compound or combined stress. An attempt is made to indicate briefly the methods used during the experiments.

" The practical results of such experiments are chiefly connected with the design of crank-shafts, and that matter is therefore discussed.

" The following systems of loading the specimen were adopted, and in each case the maximum stress on the material at elastic failure was noted :—(1) Compression only ; (2) torsion only ; (3) combined compression and torsion ; (4) tension only ; (5) combined tension and torsion. The notation for such stresses, used throughout the paper, is—for compression, $+p$; for tension, $-p$; for torsion, q.

" There are four theories of elastic failure of materials :—

"(1) *The Maximum Principal Stress Theory.*—This is known as the Rankine theory, and may be calculated by the formula—

$$\frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2} = C_1.[†]$$

" (2) *The Maximum Principal Strain Theory.*—This is known as St. Venant's theory, and may be calculated by the formula—

$$\frac{1}{E}\left\{\frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2} - \frac{1}{m}\left(\frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2}\right)\right\} = C_2.$$

" (3) *The Maximum Shear-stress Theory.*—This has been called, for mild steel, Guest's law, and may be calculated by the formula—

$$\sqrt{\frac{p^2}{4} + q^2} = C_3.$$

---

[*] This paper is printed immediately after that of Mr. Mason's on pp. 1237, &c., of Parts 3-4, 1909, *Proc. Inst. Mechanical Engineers.*

[†] See Equation V., Lecture IV., in this Vol. II., where f_t is used instead of p, f_s instead of q, and f instead of C_1.

" (4) *The Internal-friction Theory.*—This is known as Navier's theory, and may be calculated from a formula—

$$\sqrt{\frac{p^2}{4} + q^2} \sec \varphi + \frac{p}{2} \tan \varphi = C_4.$$

" The results of the author's experiments show that, for mild steel, elastic failure takes place by reason of the maximum shear stress under all conditions of loading."

Discussion.*—Mr. J. J. Guest, in opening the discussion, said the members had had the pleasure of listening to two exceedingly interesting Papers, but he wished by way of criticism to say that the authors had not exactly explained what they were aiming at. He made that remark because it was best to assume that some in an audience had not had time to digest the Papers presented, and would appreciate the discussion more fully for a sketch of the general trend of the research. He hoped the authors would therefore excuse him if he said a few words upon that point.

If a specimen in an ordinary testing-machine were pulled, it was possible to ascertain that it broke at so many pounds per square inch. Taking a member in a bridge which was in tension or compression, it could be compared directly with the result of the experiment. In the next place, take a piece of mild steel and put it into a boiler which was unstayed. There was a hoop stress round the circumference, and any rectangular part of the material (cut along the length and circumference) was subjected to a corresponding pull on its two ends. At the same time the two ends of the boiler tended to go off, and that made another pull of the material at right angles to the first pull; so that that piece of the material was subjected to tension in two directions at right angles. The ordinary testing-machine experiment gave no definite indication of the strength of the material under such circumstances, it only gave an approximation. Was it possible to ascertain what the effect of this second tension was on the resistance of the material; would it lower the strength of the material subjected to the first stress? If the material yielded at 21,000 lbs. per square inch in the testing-machine, and there were 20,000 lbs. one way, and 10,000 lbs. the other way, was that 10,000 lbs. going to make the material yield? That was the question they were trying to answer.

The important case of the crank-shaft, which Professor Smith had referred to, was similar but opposite. The second tension was then a compression. He commenced about ten or twelve years ago to try and find out the effect of the second stress, and, for the purpose, conducted a certain range of experiments. They did not answer everything, but they satisfied him that the shearing stress was the governing factor. The authors had extended that work in certain ways (see diagram).

He wished to criticise Mr. Mason's diagram on pages 1220-1 but only because he did not think the author had been quite fair in putting the figure in the manner in which he did. The author (Mr. Mason) had

---

* Students interested in this subject should study these two papers and the discussion thereon, which form a most important contribution to this subject of *Combined Stresses in Ductile Materials.* Also, see *Engineering,* Dec. 24, 1909, and *The Engineer,* Dec. 24 and 31, 1909, for abstracts, &c.

tested, the material in tension and found that it yielded at an axial stress represented by the point A in the following figure. Then he did the boiler experiment, only more so. As much tension was obtained one way as the other, and the material yielded at stresses represented by the point B. So much tension axially and the same tension circumferentially were applied and a yield obtained under equal axial and hoop tension. That told one about the boiler case, but at the point B. It was possible to put on the material a tension up to C in a testing-machine, and it had a margin of safety, or a certain factor of safety. Other tensions at right angles to it, as represented by CD, could be added on (not in a testing-machine), but it would not make the material any more likely to yield—that is, the second tension would not effect the yielding of the material under the first at all; Mr. Mason then put the material in compression, and obtained a yield-point represented by the point E (or more properly K). Then he
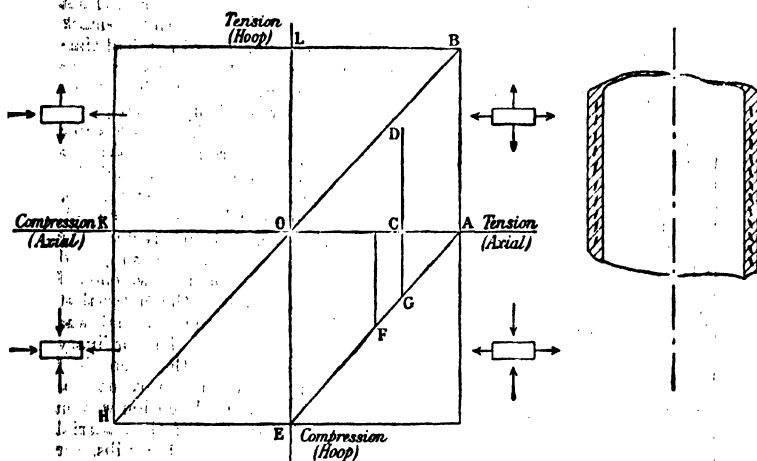
DIAGRAM OF SIMPLE AND OF COMPOUND STRESSES.

To illustrate Mr. Guest's remarks on Messrs. Mason's and Smith's papers.

DIAGRAM SHOWING DEFECTS CAUSED BY VERY HEAVY DRAUGHT OF A TUBE.

took so much tension and an equal amount of compression, but he could not in this case apply nearly as much stress. As he applied the stresses the material yielded at a point represented by F. That was the case of a shaft in torsion. With a compression instead of a tension the second stress did have an effect on the power of the material to resist.

Mr. Mason then subjected the material to two compressive stresses and found that it yielded under stresses represented by the point H. This was the exactly opposite case to the two tensions case B. The experimental determination of this point H was very difficult, owing to the high stresses involved in material which tended to buckle. This part of the work was quite new, and Mr. Mason was to be congratulated on his clever piece of work. It showed that the second compressive stress had no

effect on the capability of the material to withstand the first compressive stress.

Now Mr. Mason had obtained points A, B, F, E (or rather the corresponding point K) and H, on the diagram, but the speaker did not think that Mr. Mason should duplicate the diagram by reflection in the line B H. The point L should be reserved for hoop tension, which Mr. Mason did not experiment on. With his apparatus, Mr. Mason could obtain this point and then could justifiably mark it on the diagram, and it would afford some information as to whether the material of the tubes were isotropic. The diagram represented the whole case for a material subjected to two stresses—the important range for engineering. It would be desirable to have experiments on material subjected to three stresses, and that, he supposed, was the next step to be taken.

Turning to Professor Smith's paper, he would first reply to Professor Smith's criticism of his use of tubes. Professor Smith strongly objected to tubular specimens, but personally he thought that tubes were very suitable for some experiments. In his own experiments he had obtained a sequence of points running from A to B, and then along B towards L, and also from A to F. Mr. Mason had added the point H, and Professor Smith had made sure of the additional piece F E by obtaining a number of points along it. In attacking the subject, he had naturally selected the easiest route—the part B A F was easier than F E or the point H, the parts investigated by Professor Smith and Mr. Mason—for these latter parts involved difficulties due to the instability of the specimen.

In his own experiments, one of his difficulties was the accurate control of the pressure of the fluid inside the tubes (probably due to the small size of the secondary intensifier), but he did manage to get the points in which the tensions were equal in the two directions and went round the corner from B towards L. It, however, took some doing, and to load up using fluid pressure in apparently the way Mr. Mason had done was a difficult matter. The object of such experimental work, to find the effect of the second stress upon the yield point of a material already subjected to one stress, was represented by the determination of the line on the diagram which the research had shown to be B A F E H. And this line had been fairly well determined along B A F E and the point H also found.

Complete knowledge of the law of the yield would include the case for three stresses at right angles. In certain cases small third stresses had occurred, and he had stated in his papers his conclusion from his experiments. To investigate properly the case for three dimensions would be very difficult, though there was an easy case or two. When one considered the cases met with in actual practice, it would almost always be found that the yield occurred first on the surface, and that there, one of the three principal stresses was zero or small.

Mr. Mason and Professor Smith, working independently, had fortunately taken two different parts of the curve to experiment upon. It would have been very unfortunate if two men had put in such an immense amount of work upon exactly the same thing, and some collaboration might be suggested. If the results and curves in the papers were referred to, it would be noticed that Mr. Mason's points did not come so exactly in a line as Professor Smith's did. This was partly accounted for by the fact that Professor Smith had elaborated his apparatus, and adopted the refinement of taking account of how far the axis of stress was from the axis of the specimen. There was no doubt that it never was exactly coincident, but it could be arranged to be fairly near. However, a testing machine was

not a precision tool; it was a very accurate weighing machine. Too much was not to be expected from it.

The author stated, in the second place, "The material is not isotropic. Annealing reduces this evil, but it seems to be doubt'ul whether it entirely removes it." Personally he was very careful about the tubes he used. So far from the tubes not being isotropic, he thought they had an advantage in that respect. A tube could be made in one or two ways : either it was drawn through a die, or else it was rolled down by the Mannesmann process. Commercial considerations tended to greater drafts on the tube material, and to the use of tools until so worn that the commercial accuracy was impaired. The defects caused by overdrawing were well known, and the accuracy of the tubes used could be ascertained, and those with serious errors rejected before further work was done upon them. His tubes were smaller in diameter than Mr. Mason's, but he had never found the tubes seven-thousandths of an inch out, as Mr. Mason had the misfortune to do; they were very close indeed to being accurate. He supposed all the dies were ground. If one wished to draw a tube or wire with the least number of passes, and make a very heavy draft on it, a certain series of defects occurred in the tube or wire. The right-hand figure on p. 114 was a magnified section of the defects. Sometimes they were so big that they could be seen. Such a tube was not isotropic by any means, and it would be better not to experiment with it. It was better to make sure, first of all, that the tube was isotropic. A wire was exactly the same ; sometimes it was possible to break the wire off and get a conical section. If a piece of pianoforte wire were taken which had been drawn very hard, and if it were pulled and pushed, forming a loop, two or three times, it could l e split down the middle because of those defects ; but such tubes were not being dealt with, only carefully-made tubes. When the tube was obtained, they had, first of all, the power of applying a stress to it by a direct pull, and then there was the power of adding a circumferential stress by internal pressure. By containing these two it was possible to see whether the material was isotropic or not, because if the two interchanged and the same results were not obtained, then it was known that the material was not isotropic ; and although if the same results were obtained, then, while it was not a complete proof that the material of the tube was isotropic, it certainly gave a great degree of confidence. He did not see how that could be done on a solid specimen with the same certainty of getting the result.

If a right-hand torsion was applied to the tube, and a certain result was obtained, the results of applying a left-hand torsion would give a good hint whether the material was isotropic or not. A tension could also be put on it simultaneously with a torque, which practically had the effect of twisting the whole thing round and altering the ratios. It could be tried at a different angle. He tried all those, and found that the tubes were satisfactory.

Mr. George C. Douglas, M.I.M.E., wrote that, "it was interesting to note that the authors, from their experience, confirmed Guest's shear-stress theory of elastic failure. He believed he was correct in saying, that most of the breakdowns in gas engines had happened through the fracture of their crank-shafts, and if the author's experiments did nothing more than confirm the ideas which now obtained that gas-engine crank-shafts should be made stronger than was thought necessary a few years ago, their work would not have been in vain. But work such as theirs and work by others, as evidenced in previous papers, went much further, in so far that they provided data which could be safely used by themselves or others in

determining, say, for instance, the ultimate state of matter. However, before saying anything about what might be done in the future, he would like to ask if a note was taken of the time during which each specimen was subjected to stress, and of the temperature."

Mr. Douglas then suggested an analogy between the behaviour of mild steel under combined stresses, in which it was shown that the determining factor was "shear," and Prof. Ewing's classical experiments on the "Hysteres's Effects" produced in iron and steel during their magnetisation and demagnetisation. Finally, he explained, by aid of figures, certain experiments which he had carried out.

POWER THAT STEEL SHAFTING WILL TRANSMIT AT VARIOUS SPEEDS.

From *The Practical Engineer*, September 2, 1892. By A. G. Brown, M.E.

DIAMETERS OF SHAFTS IN INCHES.

HORSE-POWERS THEY WILL TRANSMIT.

Revs. per Minute.	1¼	1½	1¾	2	2¼	2½	3	3½	4	5	6	7	8	9	10
50	2·0	3·3	5·3	8·0	10·9	15·6	27	43	64	125	216	343	512	729	1000
60	2·3	4·0	6·4	9·6	13·1	18·8	32	51	77	150	259	412	614	875	1200
70	2·7	4·7	7·5	11·2	15·2	21·9	38	60	89	175	302	480	717	1021	1400
80	3·1	5·4	8·5	12·8	17·4	25·0	43	69	102	200	346	549	819	1166	1600
90	3·5	6·0	9·6	14·4	19·6	28·1	49	77	115	225	389	617	922	1312	1800
100	3·9	6·7	10·7	16·0	21·8	31·2	54	86	128	250	432	686	1024	1458	2000
110	4·3	7·4	11·8	17·6	23·9	34·4	59	94	141	275	475	755	1126	1604	2200
120	4·7	8·1	12·9	19·2	26·1	37·5	65	103	154	300	518	823	1229	1750	2400
130	5·1	8·7	13·9	20·8	28·3	40·6	70	111	166	325	562	892	1331	1895	2600
140	5·5	9·4	15·0	22·4	30·5	43·8	76	120	179	350	605	960	1434	2041	2800
150	5·9	10·1	16·1	24·0	32·6	46·9	81	129	192	375	648	1029	1536	2187	3000
160	6·3	10·8	17·1	25·6	34·8	50·0	86	137	205	400	691	1097	1638	2333	3200
170	6·6	11·5	18·2	27·2	37·0	53·1	92	146	218	425	734	1166	1741	2479	3400
180	7·0	12·2	19·3	28·8	39·2	56·3	97	154	230	450	778	1235	1843	2624	3600
190	7·4	12·8	20·4	30·4	41·3	59·4	103	163	243	475	821	1303	1945	2770	3800
200	7·8	13·5	21·4	32·0	43·5	62·5	108	172	256	500	864	1372	2048	2916	4000
225	8·8	15·2	24·1	36·0	49·0	70·3	122	193	288	563	972	1543	2304	3280	4500
250	9·8	16·9	26·8	40·0	54·4	78·1	135	214	320	625	1080	1715	2560	3645	5000
275	10·7	18·6	29·5	44·0	59·8	85·9	149	236	352	688	1188	1886	2816	4009	5500
300	11·7	20·3	32·2	48·0	65·3	93·7	162	257	384	750	1296	2058	3072	4374	6000
325	12·7	21·9	34·8	52·0	70·7	101·6	176	279	416	813	1404	2229	3328	4739	6500
350	13·7	23·6	37·5	56·0	76·2	109·4	189	300	448	875	1512	2401	3584	5103	7000
375	14·6	25·3	40·2	60·0	81·6	117·2	203	322	480	938	1620	2572	3840	5468	7500
400	15·6	27·0	42·9	64·0	87·0	125·0	216	343	512	1000	1728	2744	4096	5832	8000
425	16·6	28·7	45·6	68·0	92·5	132·8	230	364	544	1063	1836	2915	4352	6197	8500
450	17·6	30·4	48·2	72·0	97·9	140·6	243	386	576	1125	1944	3087	4608	6562	9000
475	18·6	32·1	50·9	76·0	103·4	148·4	257	407	603	1188	2052	3258	4864	6926	9500
500	19·5	33·7	53·6	80·0	108·8	156·2	270	429	640	1250	2160	3430	5120	7290	10000

For power of wrought-iron shafts take 70 per cent. of steel shafts of the same size.

LECTURE IV.—QUESTIONS.

1. A 10-inch shaft has a 4-inch hole run through it; what fraction of its weight is removed? To what extent is its strength in resisting torsion affected? *Ans.* 16 per cent.; 2·5 per cent. nearly.

2. A hollow shaft is 10 inches external diameter and 4 inches internal diameter; compare its strength to resist torsion with that of a solid shaft of the same weight. *Ans.* 1·26 times as strong.

3. Cylindrical bars of metal, each of 1 inch diameter, are exposed to torsion by weights applied at the end of a 12-inch lever. What would be the probable ultimate strength in the case of good specimens of wrought iron and cast iron. State the law according to which the strength of shafting increases by increasing its diameter. *Ans.* W I 800 lbs.; C I 500 lbs.

4. If a wrought-iron shaft of 1 inch diameter is broken by the torsion of a load of 800 lbs. acting at the end of a 12-inch lever, find the weight which, when applied to the end of the same lever, would break a shaft of the same material, but 3 inches in diameter. State, in general terms, the reasoning by which you arrive at the result. *Ans.* 21,600 lbs.

5. If a shaft of 3 inches diameter transmits safely 33 horse-power at 100 revolutions per minute, what size of shaft will transmit safely 20 horse-power at 150 revolutions per minute. *Ans.* 2·22 inches.

6. If 800 lbs. at the end of a 12-inch lever be a safe stress to apply to a wrought-iron bar 1 square inch in section, find the effort which a shaft 2 inches in diameter can transmit at the circumference of a pulley one foot in diameter, and making 300 revolutions per minute. Find also the horse-power transmitted. *Ans.* 8,912 lbs.; 254 H.P.

7. A shaft is of given material and given diameter, find an expression for the moment of resistance to torsion. Given the maximum stress to which the material may be subjected, find the diameter of a shaft which will transmit a given horse-power at a given number of revolutions per minute.

8. A twisting moment of 9,600 inch-pounds is sufficient to break a wrought-iron shaft of 1 inch diameter. Use 6 as a factor of safety, and hence determine what horse-power can be safely transmitted through a shaft of 3 inches diameter when running at 120 revolutions per minute. Prove the formula which you employ. *Ans.* 82·3 H.P.

9. Investigate an expression for the moment of resistance to torsion of a given cylindrical shaft when subjected to a given twisting moment. What is the maximum horse-power which could be transmitted by a shaft 3 inches in diameter when making 150 revolutions per minute, it being given that the shearing stress in the material is not to exceed 7,500 lbs. per square inch? *Ans.* 94·5 H.P.

10. If θ be the angle of twist expressed in circular measure in a length of shafting l, M the twisting moment, C the modulus of transverse elasticity, and d the diameter of the shaft, prove that—

$$\theta = \frac{10\cdot2\,M\,l}{C\,d^4}.$$

11. A horizontal bar of round iron, 1 inch diameter, 6 feet long, hinged at the ends, is subjected to equal and opposite pushing forces of 1,000 lbs. at its ends, and a load of 10 lbs. is hung at the middle so that it is

both a beam and a strut. Find the greatest stress anywhere. $E = 29 \times 10^6$ lbs. per square inch. *Ans.* 4,180 lbs. per square inch (compression).

12. Find the inside and outside diameters of a hollow steel shaft, the internal diameter being ⅝ of the external diameter. The shaft is to transmit 6,000 H.P. at 116 revolutions. Suppose the maximum twisting moment to be 1·3 times its mean value and the maximum stress allowed in the material to be 10,000 lbs. per square inch. Prove the truth of the formula which you use. *Ans.* 13·75 ins. outside diameter, 8·5 inside.

13. A shaft transmits 35 horse-power at 130 revolutions per minute; what is the twisting moment in pound feet? What is the nature of the strain and stress in the shaft? *Ans.* 1,413 lbs.

14. A wire of Siemens steel, 0·1 inch diameter, is to be twisted till it breaks. Sketch the arrangement; show how the angle of twist and the twisting moment are measured; how the results may be plotted on squared paper, and the sort of results that may be expected. In what way may a wire of twice the diameter be expected to behave. After twisting such a wire much beyond permanent set, suppose the twisting torque to be removed, in what state of internal strain might one expect to find the material?

15. A hollow tube of aluminium bronze, 1 inch diameter inside, 1¼ inches diameter outside, is to be twisted till it breaks. How would you arrange the experiment without any special testing machine? Show on squared paper what sort of results you would expect. If the modulus of rigidity of the material is $5·3 \times 10^6$ lbs. per square inch, what twisting moment will produce a twist of 0·001 radian per inch? What is now the greatest stress? *Ans.* T M = 740 lb.-ins. ; f = 3,240 lbs. per square inch.

16. Suppose that a shaft of 1 inch diameter may be safely subjected to a torque of 2,000 pound-inches, what torque will a 2¼-inch shaft safely resist? Calculate the horse-power which may be transmitted by the latter shaft if its speed is 150 revolutions per minute. How does the shear stress in a circular shaft, subjected to twisting, depend upon distance from the centre? *Ans.* T M = 22,800 lb.-ins. ; H.P. = 54·2.

17. A steel shaft is to be used to transmit power a distance of 75 feet; the twist on the whole length is not to exceed 22½°, nor the stress to exceed 8,500 lbs. per square inch: what must be the diameter of the shaft, and what H.P. can be passed through it at 135 revolutions a minute? (Modulus of transverse elasticity = 12×10^6 lbs. per square inch.) *Ans.* Diam. = 3·25 ins. ; H.P. = 123.

18. Compare the strength of two cylindrical shafts, subjected to pure torsion, if their diameters are 1¾ and 2¼ inches respectively. Assuming that the cost of such shafting is directly proportional to its weight, what would be the relative costs? *Ans.* 1 : 1·65.

19. Obtain a formula for the diameter of a shaft, giving the twisting moment to which it is exposed, and the maximum stress allowed in the material :—A shaft 15 inches in diameter is exposed to a twisting moment of 227,800 foot-pounds, and to a maximum bending moment of 43 foot-tons: find (*a*) the maximum stress set up in the material, (*b*) the twisting moment required to produce the same stress if there were no bending moment acting with it.

20. The two halves of a flange coupling are fastened together by four round wrought-iron bolts. The diameter of the bolt circle is 10 inches, and the shaft transmits 120 H.P. when making 120 revolutions per minute. Assuming a suitable sheer stress for the bolts, find their diameter. *Ans.* 1 inch.

21. Find the diameter of the propeller shaft for a ship, each engine of which develops 10,000 H.P. when going at 120 revolutions per minute. You may assume that the shaft is subjected to pure torsion, and that the safe shear stress of the material is 5 tons per square inch.

22. The screw shaft of a high-speed vessel is 6 inches external and 3 inches internal diameter, and rotates 400 times a minute. If the intensity of stress is limited to 5 tons per square inch, find the maximum horse-power that can be safely transmitted, the shaft being supposed subjected to twisting only. If the length of the shaft between the thrust block and the screw be 60 feet, find the angle of torsion of the shaft, the modulus of rigidity being 4,500 tons per square inch. *Ans.* H.P. = 2,830, θ = 15·3°.

23. In an overhanging crank, the crank-arm radius is 16 inches and the distance between the centre of the crank-pin and the centre of the near crank-shaft bearing is 12 inches. When the connecting-rod is at right angles to the crank, the thrust along the rod is 5,000 lbs. Estimate the maximum tensile and shearing stresses in the crank-shaft, the diameter of the crank-shaft being 5 inches. *Ans.* Shear stress = 3,260 lbs. per square inch; tensile 4,890 lbs. per square inch.

1. A turbine is connected to a dynamo, placed vertically above it, by a shaft 2 feet in diameter made of steel plate $\frac{5}{8}$ inch thick. Calculate the diameter of a solid shaft of the same material to transmit the same power at the same speed with the same maximum skin stress, due to twist. Find the relative weights. *Ans.* Diameter solid shaft = 13·9 inches. Relative weights ·304 : 1.

2. A mild steel shaft transmits 100 H.P. at 120 revolutions per minute, and is subjected to a maximum bending moment of 2,000 lb.-feet. Find a suitable diameter if the maximum resultant stress is not to exceed 6,000 lbs. per square inch. *Ans.* 4·5 inches.

3. A shaft, 3 inches in diameter, running at 250 revolutions per minute, transmits 50 H.P. Find the maximum stress and the twist of the shaft in degrees in a length of 100 feet. The rigidity modulus is 12,000,000 lbs. per square inch. *Ans.* 2,400 lbs. per square inch. 9° per 100 feet.

4. Prove the common formula for the resultant stress in a circular shaft subject to combined bending and twisting.

5. A shaft S is driven at a speed of 800 revolutions per minute by means of an engine of 300 H.P., the speed of piston being 500 feet per minute. The connection between the shaft S and that of the engine is effected by a belt which embraces a pulley of 8 inches diameter on the shaft S. Neglecting losses by friction, estimate the mean pull and thrust of the piston-rod, the torsion of the shaft S, and the tension of the belt, assuming that of the slacker side to be one-third of that of the tighter.

6. A shaft 20 feet long has to transmit a twisting movement of 1,000 lb.-feet when revolving at 200 revolutions per minute. Find the required diameter so that the shaft will not twist more than 2 degrees, and that the maximum stress shall not exceed 5 tons per square inch. *Ans.* Diameter = 2·9 inches for stiffness ; 1·8 inch for strength.

7. A marine engine whose mechanical efficiency is 88 per cent. indicates 1,000 H.P., and runs at 80 revolutions per minute. The speed of the ship is 12 knots. What size would you make the propeller shaft for this engine if solid, allowing 4 tons per square inch ? What would be the direct compression stress in it between the propeller and the thrust block, and what would be the stress due to the twisting moment between the engine and the thrust block ? *Ans.* 8 inches ; ·23 ; 4·5 tons per square inch.

8. Two steel shafts 6 inches in diameter have to be coupled together by an ordinary coupling. The diameter of the pitch circle of the bolts is 13·5 inches, and the torque is 6,000 foot-lbs. Calculate the number and size of the bolts (steel), and make a hand sketch of the coupling.

9. What conditions determine the dimensions of the crank pin of a simple overhung crank ?

10. The crank shaft for a gas engine should be much larger in diameter than the shaft of a steam engine of the same power and speed. Why is this ?

11. A wrought-iron crank shaft, 6 inches diameter and 20 feet long, has a load applied to the crank which twists the ends through an angle of 2° ; taking the coefficient of transverse elasticity as 9,000,000 lbs. per square

inch, what is the stress at the extreme fibre? *Ans.* 3,930 lbs. per square inch.

12. A shaft has to transmit 50 H.P. at 300 revolutions per minute, find the required diameter of the shaft, so that the extreme fibre stress may not exceed 5 tons per square inch. *Ans.* 1·68 inch.

13. Determine an expression for the amount of twist in a circular shaft under a given twisting moment. A steel shaft is 3 inches in diameter and 60 feet long; find the angle in degrees through which it will be twisted if subjected at one end to a twisting moment of 20,000 lb.-inches.

14. What horse-power can be transmitted by a steel shaft 4 inches in diameter when running at 300 revolutions per minute; the stress in the extreme fibre to be 6 tons per square inch?

LECTURE V.

STRENGTH AND ELASTICITY OF MATERIALS.
TESTING AND TESTING MACHINES.

CONTENTS.—Short History of Mechanical Testing Machines – Advantages of Laboratory Testing to Young Engineers—Mechanical *plus* Chemical and Microscopic Tests of Engineering Materials — Different Mechanical Tests—Necessity for a Careful Prior Investigation of the Kind of Stresses to which the Structure may be Subjected—Investigations of Accidents and Breakages—Prof. Barr's Wire-Testing Machine —Line Diagrams of Three Leading Types of Testing Machines—Thirty-Ton Single-Lever Vertical Testing Machine, with Autographic Recorder—Fifty-Ton Single-Lever Testing Machine—Buckton & Co.'s 300-Ton Universal Testing Machine for Full-sized Structural Members —Equipment of this Machine for Different Kinds of Tests—A Convenient Method of Obtaining and Regulating the Pressure and Flow of Water to the Ram of a Testing Machine—Proportions of Test-Bars and their Attachment in Tensile Tests — Transverse Bar - Testing Machines—Cross Bending, &c. — Admiralty Rules for Testing Materials for Machinery—French Methods of Testing Materials of Construction—Publications Dealing with Engineering Tests and Materials—Questions.

Short History of Mechanical Testing Machines. — Instead of actually loading a specimen direct with dead weights, we may save both time and labour by using a testing machine which measures by some more or less convenient "advantage" or "moment" the stress or load applied to a specimen. The greater the load required to test the specimen, the greater will be the saving between direct loading and the application of the desired stress in an up-to-date accurate testing machine, which is simply an ingeniously devised and refined weighing apparatus.

Naturally, the most primitive form of testing machine was to fix one end of the specimen to a rigid foundation, then to attach the other end of the specimen to the end of the shorter arm of a balancing lever, and put known weights into the scale pan hung from the end of the longer arm until the desired effect was produced on the specimen.

In 1813 an improvement upon this simple arrangement was made and introduced into the Royal Dockyard at Woolwich by the well-known inventor of the hydraulic press, Mr. Bramah of Pimlico, London, for the purpose of testing the chain cables for mooring ships, which were then taking the place of rope hawsers. This machine had a fixed hydraulic cylinder with its ram-head hooked on to one end of the chain to be tested. The other end of the chain was attached to levers with balancing weights, whereby the stresses produced upon the chain by the pressure of water on the ram were indicated.

In 1852 Ludwig Werder, of Nürnberg, made a machine for applying both tension and compression stresses to test specimens.

From April, 1858, to September, 1861, Mr. David Kirkaldy, head draughtsman to Robert Napier & Sons, Engineers and Shipbuilders, Glasgow, carried out a most extensive and careful set of tests and other experiments upon "homo" and "puddled" steels for marine steam boilers and machinery, as well as upon the iron plates and angle irons of H.M.

armour cased ships "Black Prince" and "Hector"; and The Institution of Engineers and Shipbuilders in Scotland awarded him their Prize Gold Medal for his communication "Experiments on Iron and Steel." *

From the beginning of 1862 to June, 1864, Mr. David Kirkaldy devoted two and a half years to the designing, drawing, and patenting of a specially large new form of horizontal testing machine, which he then entrusted to Greenwood & Batley, Leeds, to make under his supervision. This machine was finally set to work at The Grove, 99 Southwark Street, London, on the 1st January, 1866, for the purpose of public and private testing, and is still doing good work to the present day. By this machine, columns of any length up to 21 feet 6 inches by 2 feet by 2 feet 8 inches can be tested up to 270 tons, which gives some idea of its size and capabilities. It was arranged for tension, compression, and deflection tests.† The author had the pleasure of seeing this machine at work in 1873, and of having its construction, action, and uses explained to him by the designer.

There can be no doubt, that to Mr. David Kirkaldy, M.Inst.C.E., &c., belongs the credit of first designing and setting to work an accurate, large testing machine, as well as of carrying out thousands of different reliable experiments with it for various governments, firms, and persons. The more important and larger sizes of testing machines used in many works and laboratories in Europe have been based upon the leading principle of this machine—viz., that of applying the load to the test-piece by water pressure in a hydraulic cylinder and of measuring the load by a weight.

From 1860 to 1870, Herr Wöhler carried out many experiments on iron and steel, and devised several testing machines for subjecting specimens to oft-repeated stresses, or fatigue.

In 1872-73, Lord Kelvin's Hydrostatic Testing Machine was first used for testing thousands of miles of the sheathing wires for submarine cables. (See the author's *Manual of Applied Mechanics* for figure and description.)

In 1879, Mr. A. H. Emery built and started a testing machine at the Watertown Arsenal, U.S.A., with a peculiar diaphragm-piston or press cylinder, &c., which could do all the different tests of Kirkaldy's machine, and subject columns 30 feet long by 2 feet 6 inches by 2 feet 6 inches to at least the same thrust, or 357 imperial tons.‡

In August, 1882, Mr. J. H. Wicksteed, of Leeds, described in his paper, read before the Institution of Mechanical Engineers, his special form of Single-Lever Testing Machine; and, in February, 1886, before the same Institution, his "Autographic Test-Recording Apparatus" for use with his vertical or horizontal testing machines.

In 1894, the great Charlottenberg testing machine was installed. It tests up to 492 imperial tons, and takes in columns 49 feet long by 2 feet 7 inches by 2 feet 7 inches.

---

* See *Trans. Inst. E. and S.*, 1862-63, vol. vi., p. 27, for Mr. Kirkaldy's paper with conclusions and discussion; also vol. vii., p. 134, for an account of the presentation of the medal.

† All students interested in Mechanical Testing and Testing Machines should try and see the large book by W. G. Kirkaldy on *Strength and Properties of Materials with Description of the System of Testing*, published by Sampson Low, Marston, Searle & Rivington, Limited, London, since they will find therein a large quantity of useful data, with illustrations and description of the above-mentioned 1866 Kirkaldy Testing Machine.

‡ See vol. lxxxviii., *Proc. Inst. C.E.*, p. 14, for Prof. Kennedy's paper on "Engineering Laboratories," where further historical and other data will be found.

In 1903, Mr. J. H. Wicksteed, President Inst. M.E., designed, and his firm, Joshua Buckton & Co., Limited, of Leeds, made and installed a 300-ton Universal Testing Machine at the Testing Laboratory of the Conservatoire des Arts et Métiers. This machine is 130 feet in length and 120 tons in weight. It will take in columns 88 feet long by 3 feet 3 inches (each way), and tension members of the same length and diameter. *

Advantages of Laboratory Testing to Young Engineers.—

During an ordinary apprenticeship in a works or pupilage in a civil engineer's office, the average young engineer does neither have much opportunity of studying scientifically the physical, chemical, and microscopic (or minute structural) properties of the metals with which he may be brought into contact, nor their strengths and elasticities under different kinds of stresses. Moreover, he has seldom full access to specifications wherein the various tests to be applied to the work in hand are detailed. True, he can attend evening science and technological lectures, or study at home under a system of "tuition by correspondence." But, neither his apprenticeship nor his evening studies, however well performed, will afford him the same comprehensive grasp, or give him the same intimate and useful insight into the properties, behaviour, and efficiencies of engineering materials and machines as a thoroughly good set of day engineering laboratory courses. There, he can be taught not only how to arrange for, but to make by himself, or in conjunction with other students, accurate measurements, tests, and a variety of experiments.

Professor A. B. W. Kennedy was the first person to start, in 1878, at University College, London, such an Engineering Laboratory; and now, most of our Universities and larger Technical Colleges possess imitations of, and even improvements upon, what he initiated. Consequently, young engineers who can spare the time and the money may attend and be greatly benefited by such practical Laboratory courses.

This slight diversion here on the author's part from his otherwise strict adhesion to explanatory diagrams, calculations, and technical explanations, is partly due to the fact, that he is indebted to Profs. Kennedy and Ewing, as well as to the Institution of Civil Engineers, for permission to reproduce in this lecture a number of figures with explanations of the actual testing machines used by them and their students in their laboratories. A careful study of these descriptions, as well as of the other testing machines for works, with the references to papers and books, will no doubt prepare those who have served an apprenticeship to mechanical engineering to make tests with similar machines. They will also help pupils who have not yet done so, but who intend to enter upon a laboratory course or who have begun the same, to understand the actual machines and the different tests far better than if they had previously never read about them.

---

* See Mr. J. H. Wicksteed's 1904 B.A. paper on "A Universal Testing Machine of 300 tons for Full-sized Structural Members," and description of this machine in this lecture.

Mechanical plus Chemical and Microscopic Tests of Engineering Materials. It is now generally recognised, that prior to the designing and during the manufacture of all important new engineering structures, specimens of the proposed and of the actual materials should be subjected to certain tests and investigations which shall, as far as possible, imitate and prove their fitness to withstand the very stresses and strains, as well as the other affecting influences, to which the different parts of the structure may have to be subjected, during their useful existence.

In many cases, mere mechanical tests will not reveal the complete all-round knowledge of the fitness of a specimen - or even of a complete member—to withstand in the best manner possible, and for the longest time, the various influences which may be brought to bear upon engineering materials during their lifetime. Hence, it is frequently necessary, that these mechanical tests—however skilfully applied and their results accurately measured, tabulated, and considered—should be accompanied by exact chemical analysis, as well as by careful microscopic observations with photo-micrographs. *

Micrographs.—For example, the researches of Professor J. O. Arnold,† at the University College, Sheffield, have revealed, that almost invariably *small* steel castings exhibited in the first stage of their manufacture the Widmannstätten figures, when the carbon in the steel was near the semi - saturation point of steel — viz., 0·45 per cent. His researches involved a close investigation of the influence of mass, and hence the experimental castings varied from 28 lbs. to 2 tons. In the heavier castings the particular Widmannstätten figures already mentioned are seldom to be found, since the slow cooling of the mass exerted an influence similar to that of annealing, which operation causes a change in the structure so profound as almost always to destroy these figures.

The Steel Casting Micrograph Figure.—The structure of the metal from a 30-lb. casting is shown by the upper half-section of the micrograph figure. It exhibited only two of the constituents since the original magnification of 66 diameters was too low to reveal the third and fourth constituents—viz., the sulphides of manganese and of iron—which were present in minute quantities.

The dark etching constituent is pearlite (21Fe + Fe₃C). Its colour is due to the liberation during etching of an automatic stain composed of that dark, carbonaceous colouring matter upon which the well-known carbon colour test depends. The pale constituent is ferrite, or nearly pure iron, and has assumed that crystalline structure characteristic of the Widmannstätten figures.

---

* In order that students may the more thoroughly grasp the importance of carrying out their investigations of metals in a complete manner, they should study "Naval Accidents," by Thomas Andrews, F.R.S., in *Engineering*, p. 737, December 2, 1904, *et seq.*

† I am indebted to Prof. Arnold for kindly presenting me with the original photo-micrographs of 66 diameters, from which my publishers reproduced the accompanying plate (Fig. 1) to half size, or 33 diameters, as well as drawings of the two test-pieces, and to *Nature* of November 10, 1904, p. 32, for the data by Prof. Arnold and Mr. A. M'William.

AS CAST. 43° BROKE.

AS CAST

AFTER ANNEALING

AFTER ANNEALING. 180° UNBROKEN.

Fig. 1. Test Specimens of Cast Steel.
Made by Professor J. O. Arnold and A. M'William at the
University College, Sheffield, 1904.

Chemical Analysis.—The mean analyses of drillings taken from a portion of the casting were as follows :—

Carbon,	0·39	per cent.
Silicon,	0·08	,,
Manganese,	0·03	,,
Sulphur,	0·03	,,
Phosphorus,	0·02	,,
Aluminium.	0·03	,,
Iron by difference,	99·42	,,
Total,	100·00	,,

The Annealed Steel Casting Micrograph.—The lower half-section delineates the very different structure of the cast steel after careful annealing. The result shows a total re-arrangement of the pattern presented by the ferrite and pearlite, and a consequent elimination of the above-mentioned figures.

Mechanical Tests of the Cast and Annealed Steels.—The very complete change in structure due to annealing was also accompanied by a great change in the mechanical properties of the steel. This change will be seen by comparing the results of bending tests made before and after annealing, upon two bars each 10 inches long by ¾ inch diameter from the casting. The metal as cast snapped sharply after bending through an angle of 30° over a radius of ⅜ inch; whereas, the annealed bar bent through an angle of 180° over the same radius without any signs of fracture. The following results were obtained from this steel casting :—

Upper and Lower Halves of Plate, Fig. 1.	Specific Gravity.	Elastic Limit. Tons per Sq. In.	Maximum Stress. Tons per Sq In.	Elongation on 2 Ins.	Reduced Area.	Compression at 100 Tons per Sq. In.
Casting,	7·956	13·33	24·62	8·4 °/₀	12·3 °/₀	57·3 °/₀
Annealed,	7·978	12·21	23·50	14·0 °/₀	16·0 V₀	62·3 °/₀

This particular steel casting was not intended for commercial purposes; but the results as stated are sufficient to show, that a thorough investigation into the properties of metals for engineering structures should be accompanied by exact chemical analysis and by precise metallographic reproductions of the structures, before and after any special treatment, such as annealing or tempering in water or oil, &c., and even before and after the mechanical tests.*

---

* Students who are specially interested in the various phases of this subject should study the several reports of "The Alloys Research Committee," as started and maintained by The Institution of Mechanical Engineers, London, from 1891 to 1905. A synopsis of five reports will be found in the General Index to the *Proceedings*, 1885 to 1900, of that Institution, as issued in November, 1904, under the heading "Alloys Research."

Different Mechanical Tests.*—Messrs. Seaton and Jude in their paper on "Impact Tests" remark, that –"At present everybody makes tension tests and trusts more or less to them alone"!

The tension test is no doubt a good and sufficient one when the predominant stress is that of pure static tension or steady load as in boilers, buildings and tanks, &c.; or, where the structural work may be subject to recurrent loads of *one kind* (*with intervals of rest*), in addition to the steady load due to its own weight as in the several members of bridges, &c. But, in the case of structures subjected to rapidly-repeated loads of one kind, all more or less suddenly applied, as with bolts, studs, rails, &c.; or, structures subjected to alternating loads, such as in the fixed and moving parts of machinery in general, and many parts of a ship, the simple tension test is not sufficient.

But, the following tests can be made on most pieces of metal and on steel in particular :—

1. Tension and elongation.
2. Compression.
3. Transverse bar tests.
4. Cross bending, or folding, or doubling cold or hot.
5. Shearing, boring, drifting, punching; hammering to a point or edge.
6. Fatigue of metals by gradual reversal of stress either by bending in one plane or by oscillations, as adopted by Wöhler and others.
7. Fatigue of metals by reversals of the mean stress, as carried out by Prof. Osborne Reynolds and J. H. Smith, M.Sc.
8. Impact on unnotched and on notched bars.   Drop or falling tests.
9. Torsion.
10. Hardness, temper, and brittleness.

The above-mentioned authors naturally ask the question, "Which of the tests in such a long list is really a true universal gauge of the suitability of (say) a piece of steel for any purpose it may be put to?" And, as a help to answer this question, they give the following interesting table of the different kinds of stresses in the steel parts of an up-to-date (reciprocating) steam engine of moderate size:—

---

* See the paper read before The Institution of Mechanical Engineers, on November 18, 1904, by A. E. Seaton of London and A. Jude of Birmingham—viz., "Impact Tests on the Wrought Steels of Commerce," and the discussion.   Also, the reports and articles on the same in *The Engineer* and *Engineering* of November 25, 1904, *et. seq.*   Students may also refer to Appendices D to M at the end of Mr. A. E. Seaton's *Manual of Marine Engineering*, 15th or later editions, as published by Charles Griffin & Co., for the Admiralty, Board of Trade, Lloyds, British Corporation, and Bureau Veritas Rules for Boiler Shells, Plates, Stays, Shafting, and their respective tests.   The British "*Admiralty Rules for Testing Materials for Machinery*," and French "*Methods of Testing Materials of Construction*," will be found at the end of this lecture.   Books containing all the rules and regulations of each of these authorities may be obtained by direct application to their secretaries.   These books are to be found in the libraries of most engineering institutions.

DIFFERENT KINDS OF STRESSES TO WHICH THE STEEL PARTS OF A RECIPRO-
CATING STEAM ENGINE ARE SUBJECTED AND THEIR PERCENTAGE VALUES.

Constant tension, ...	3·91	per cent.
Constant tension and compression (range from 0 to a max.),	1·30	,,
Constant tension and shock, ...	48·80	,,
Alternating tension and compression with shock, . .	2·81	,,
Repeated tension (from a constant to a max.) with shock,	36·00	,,
Miscellaneous and doubtful, ...	7·17	,,
Total, ...	100·00	,,

"It will therefore be seen, that 87·6 per cent. of the whole of the engine's stresses are more or less due to shock, whilst pure tension stresses form an insignificant percentage of the total stress." They argue that, if various other machines be examined, it will be found that 9 out of 10 are working under similar conditions.

Necessity for a Careful Prior Investigation of the Kind of Stresses to which Structures may be Subjected.—The above short quotations and abstracts are chiefly made here, to draw the student's attention to this important paper, and to the fact, that it is the duty of the engineer to consider most carefully whilst he is designing any fixed engineering structure or moving machine, what are the various and the chief kinds of stresses to which the structure or machine will be subjected during their existence. Further, to determine and specify clearly how each specimen test-piece for each important member should be tested, so that he may ascertain to the best advantage its suitability for its future duties.

Investigations of Accidents or Breakages of Materials.— When making investigations of accidents or causes which have led to the breakage of materials, the engineer should get or produce—

1. A complete set of scale and sized drawings of the parts in question.
2. Photographs of these parts taken from different points of view, and, if possible, before any alterations have been made.
3. As logical a statement as can be gathered of the conditions under which the breakage or accident occurred.
4. Accurate chemical analysis of the parts near the fracture.
5. Large photo-micrographs of these parts to show the structure of the materials.
6. Carefully machined test-pieces (at least in two right-angled directions) taken from different places near the fracture and elsewhere.
7. Such tests of these machined pieces as will most likely prove their qualities of resisting maximum stresses under fair conditions.
8. He should then draw up a concise, clear, illustrated description of the whole of the circumstances and investigation.

The Order of this Subject.—In this and the following lectures, the order of all the previously mentioned tests will be adhered to as far as possible, beginning with a testing machine for the elongation and breaking stress of wires, and then proceeding to larger machines for making tension, compression and transverse bending tests, &c. Admiralty rules for and French methods of testing, and stress-strain diagram recording apparatus will be illustrated and described with results, as well as how Young's modulus of elasticity is obtained for different cases, and, finally, the strength of struts and columns will be considered.

WIRE

PEN

WIRE CLIP
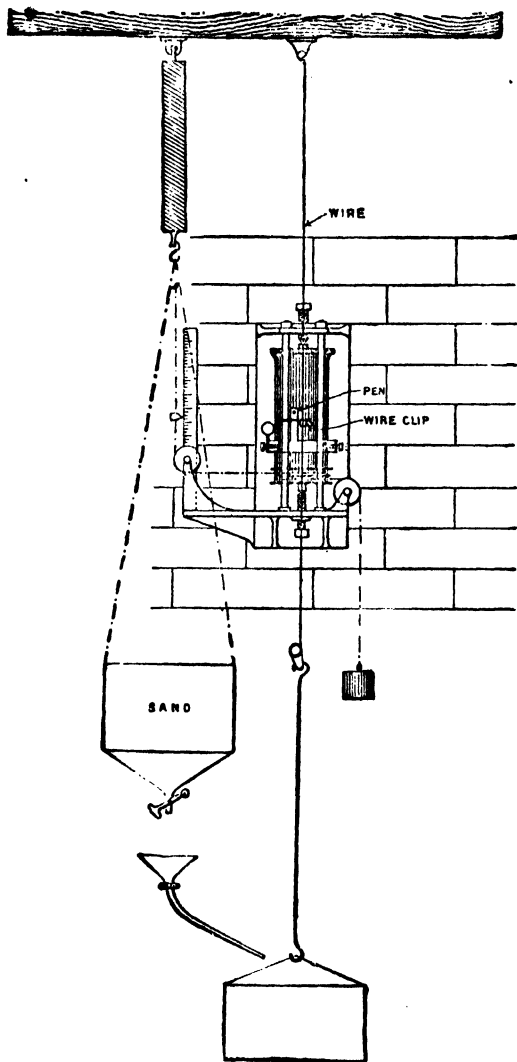
SAND

FIG. 2.—PROF. BARR'S WIRE-TESTING MACHINE.

Prof. Barr's Wire-Testing Machine.*—Fig. 2 illustrates clearly the general arrangement and method of performing the test, viz. :—By admitting sand from the sand-box through a filler to the box attached to the end of the test-piece. The weight of sand allowed to pass is registered by the pointer on the graduated vertical scale.

If test-pieces of annealed wires differed from an ordinary specimen, it would only be in their having greater homogeneity. Therefore wire testing has this great advantage, that a large number of practically identical specimens can be obtained and tested. Hence, in testing two wires cut from one coil with the small machine shown by Fig. 2, if the two diagrams are made upon one paper with a fine pen under like conditions, it will be found, that the lines agree so closely for a considerable part of their length as to be undistinguishable from each other.

Line Diagrams of Three Leading Types of Testing Machines.† —In the previous article on the history of testing machines it should have been mentioned, that for small machines up to 5, 10, or even 15 tons stress it is usual to employ screw and gearing instead of the hydraulic ram. This can be done without any loss of accuracy, and for ordinary laboratory work such sizes and method of applying the load are fully as convenient.

The following three diagrams are simply intended to illustrate the principle of each machine to the student and not details or proportions :—
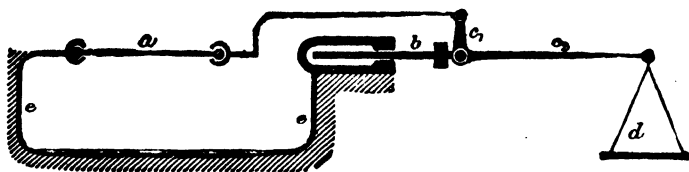
FIG. 3.—LINE DIAGRAM OF A WERDER TESTING MACHINE.
As used by Bauschinger at Munich in 1871.

Werder Machine.—Here the test-piece a is held at one end by a grip on the frame of the machine e, and the other end is pulled by a connecting-rod from the short arm c_1 of a knee-lever, whilst a scale pan d hangs from the outer end of the long arm c_2 of the lever. The central fulcrum of the lever rests upon the end of the ram b, so that the whole of the measuring apparatus moves along coincident with the motion of the ram head and the extension of the test-piece. The arm c_2 is kept horizontal by aid of a spirit level.

University College, London, Testing Machine.—This type of machine was designed and made by Greenwood & Batley in 1878 to meet Prof. Kennedy's requirements in his laboratory at University College, London. In principle it is the same as the much larger horizontal machine previously referred to as designed by Mr. Kirkaldy. It consists of a

---

* This wire-testing machine is used in the Watt Engineering Laboratory at Glasgow University. For a figure and description of Lord Kelvin's hydrostatic wire-testing machine, see the author's *Manual of Applied Mechanics.*

†Figs. 3, 4, 5, 8, 9, and 10 are from Prof. Kennedy's paper on "Engineering Laboratories," by his kind permission, and by the favour of the Institution of Civil Engineers. See *Proc. Inst. C.E.*, vol. lxxxviii.

hydraulic ram b, connected directly to one end of the test-piece a, whose other end is attached to the knee-lever f_1 f_2 (5 to 1) and then by a link to a steelyard lever c_2 c_1 (20 to 1).   The total leverage is therefore 100 to 1.
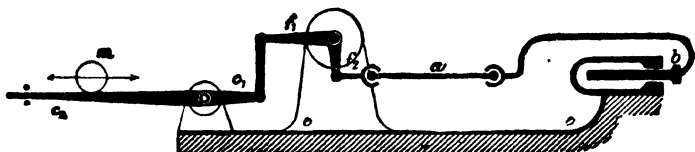
FIG. 4.—LINE DIAGRAM OF THE TESTING MACHINE AT THE
UNIVERSITY COLLEGE, LONDON.

As made by Greenwood & Batley, Leeds, in 1878.

The load is applied by the ram b and measured by the position of the poise-weight m on the steelyard.   The carriage and its hanger weigh 50 lbs. by themselves, but the poise-weight m is variable by the addition or subtraction of 50 lbs. at a time up to 1,000 lbs.

Wicksteed's Machine.—This vertical machine has a single straight lever c_2, c_1 (50 to 1) placed horizontally on the top of the column e of the machine.   A movable poise-weight m measures the load applied by the
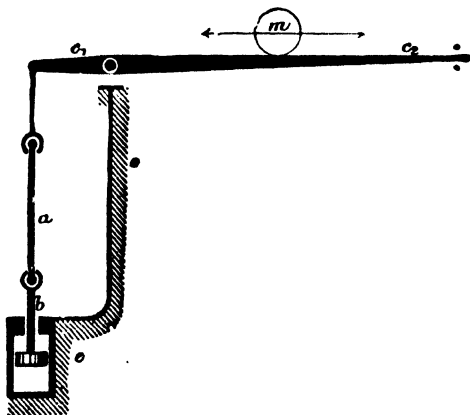
FIG. 5.—LINE DIAGRAM OF A WICKSTEED VERTICAL SINGLE-LEVER
TESTING MACHINE.   As made by Buckton & Co., Leeds, 1882.

ram b to the test-piece a.   Thus, when a certain poise-weight is run out to its stop at the end of the lever c_2, it balances 50 times the stress put upon the test-piece a by the ram b.   The poise-weight m runs on a four-wheel carriage, which is moved along the steelyard lever by a central screw, that is generally turned by belt power in the larger and by hand and a lever in the smaller machines.   The pull on the test-piece is read by aid of a vernier attached to the poise-weight, whilst the scale proper is fixed to the steelyard.   This type of machine is generally speaking simpler and better appreciated than the two former types.

Thirty-Ton Single-Lever Vertical Testing Machine, with Autographic Recorder.—*Objects to be attained with the Machine.*—The machine illustrated by Fig. 6 is for testing the strength of different materials in tension, bending, and compression. It is made by Greenwood & Batley, Ltd., Leeds, to test up to 30 tons, and to admit test-pieces 3 feet in length for tension, 24 inches long for compression, also beams 14 inches deep by 7 inches wide for bending. The supports are made adjustable from 6 inches to 5 feet apart (see Fig. 5 for the line diagram).
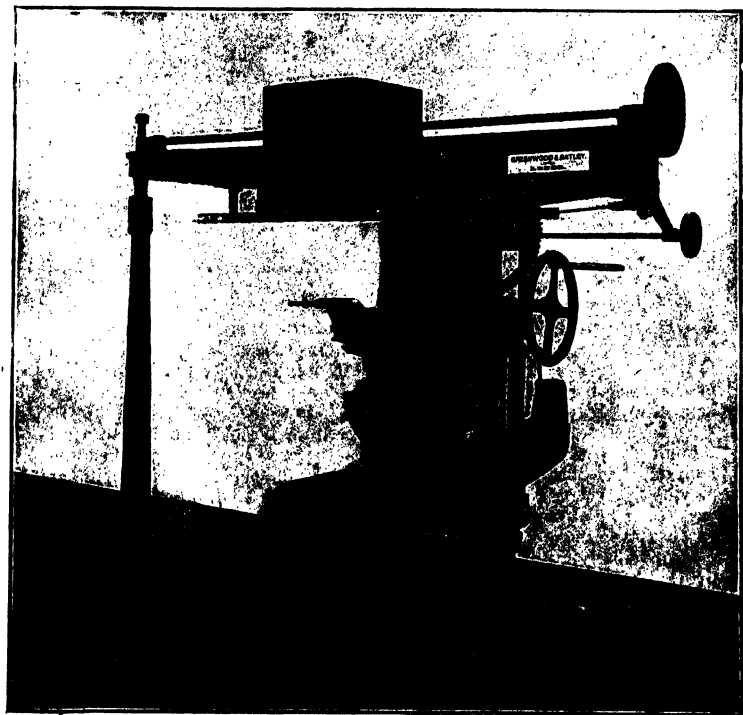
FIG. 6.—THIRTY-TON SINGLE-LEVER VERTICAL TESTING MACHINE.

By Greenwood & Batley, Ltd., Leeds.

Description of the Machine.—It consists of a heavy, vertical cast-iron support, upon which is mounted the weigh beam, with its travelling weight of 1 ton. This travelling weight is moved along the beam by means of a quick threaded screw, which may be rotated either by hand or by hydraulic power. The beam carries a graduated scale to read direct in tons and fractions of a ton. The weigh beam and die holders rest on hardened steel knife edges. The lower die holder is attached to

an adjustable crosshead connected to the hydraulic ram crosshead by two steel screws, and is balanced by a weight and spring beam, shown at the bottom of the left-hand side of the vertical support. This cylindrical balance weight serves to minimise the shock when the test-piece breaks.

The testing machine is provided with different forms of clips for holding the various lengths of test-pieces, as well as an autographic recorder, shown on the side of the vertical column, for plotting mechanically a stress-strain diagram of the test-piece.

Fifty-Ton Single-Lever Vertical Testing Machine* (See Frontis-plate).

—A convenient form of testing machine for use in laboratories is illustrated by the frontisplate of this edition. Here, a single self-balanced lever, with a single heavy travelling poise-weight, is the means for measuring the load applied to the specimen by a direct screw or by the hydraulic ram. This testing machine consists of a strong upright column to which the cylinder of the hydraulic ram is attached near the foot. A long lever or weigh-beam rests by a knife-edge on the top of the column and carries a travelling poise-weight. This travelling weight weighs 1 ton and carries a vernier, by means of which its position is read against a divided scale on the beam. The test-piece is placed vertically, with its lower end secured in a crosshead, which is pulled downwards by the hydraulic ram underneath it. The upper end of the test-piece is secured by a shackle, which hangs from an inverted knife-edge on the beam. The beam oscillates about its knife-edge, which is placed at a short distance to the left of the first one, and is supported by a fulcrum plate on the top of the upright column. The travelling poise-weight is moved by means of a screw concealed within the beam. This screw receives its motion through spur-wheels from a parallel shaft provided with a Hooke's joint in the axis of oscillation of the beam. This shaft is turned either by a hand-wheel or by a power-driven counter-shaft.

The cylindrical counterpoise, which is seen projecting behind the central column and near the floor level, serves to force up the ram when the hydraulic pressure is relieved. The pressure in the hydraulic cylinder may be applied either by means of a belt-driven, hydraulic, screw com-pressor, or by an accumulator, or, better still, by a hydraulic intensifier. In the latter case, the work is done by admitting water from a low-pressure supply main behind a large piston, which forces forward a small hydraulic plunger and produces an intense pressure in the fluid upon which the small plunger acts, from which it is in turn transmitted by a pipe to the straining ram of the testing machine. This arrangement has the advantage of allowing the load to be applied to the test-piece without shock, and at as quick or as slow a rate as may be desired. The regulation of the rate of application of the load is obtained by means of a throttle valve, through which water from the low-pressure supply main has to pass on its way to the large cylinder of the intensifier.

Arrangements for Different Tests.—This testing machine is arranged for compressive tests by admitting a test-piece 24 inches long ; for deflection tests, with a maximum length of 60 inches ; shearing, 1⅛ inch diameter ; torsion, for a load of 50,000 lbs. at 1 inch-moment, with a maximum length for observation of 10 inches, as well as for tension tests.

For compression tests, the upper shackle is connected to a platform by means of four columns in the shape of a cross-beam.

---

* See *Proc. Inst. Mech. Engs.*, 1882, for paper on "A Single-Lever Testing Machine," by J. Hartley Wicksteed, of Leeds.

This cross-beam hangs below the cross-head, which is pulled down by the hydraulic ram. It will be seen, that the arrangement is two stirrups linked with one another, but having one of them inverted, so that, when the two are acted upon, they pull against each other and cause a block of material, or test-piece placed between them, to be compressed.

For tests in bending, the stirrup or beam which hangs by columns from the upper shackle is made between 4 and 5 feet long, and carries supports at its ends for the ends of the test-piece, while the cross-head presses down on the middle of the piece (see figure). In both instances, the force which is exerted is measured by means of the weigh-beam and travelling poise-weight, in the same way as shown for the tension tests.

For torsion tests, the worm and worm-wheel shown in the figure at the top of the upright serve to twist the test-piece, one end of which is secured in the axle of the worm-wheel, while the other end is secured in a socket which projects from the side of the weigh-beam. The axis of the test-piece under torsion is in the same line as the knife-edge about which the beam is free to oscillate.

Knife-edges in Testing Machines.—Each of the knife-edges in the weigh-beam of Buckton & Coy.'s testing machines is the edge of a square cut-bar of steel, and they are made long enough to prevent the load on them from exceeding 5 tons to the linear inch.

Buckton & Co.'s 300-Ton Universal Testing Machine for Full-sized Structural Members.[*]

—This machine is composed of three principal parts:—(1) The apparatus which applies the load to the test-piece, (2) the apparatus which measures the load, and (3) the frame or bed.

There are two great advantages in the arrangement of this testing machine:—*First*, the bed is *movable*, instead of being fixed as in ordinary machines ; and, *second*, the weighing levers are placed at the same end as the stressing cylinder, and can be worked by one man. These levers are mounted upon this stationary cylinder, instead of being carried on the moving ram, as in the Werder machine. It is this combination, that gives the machine the facility which it possesses for testing such a great variety of sizes, and for subjecting full-sized members to compression or tension without the necessity of re-arranging the machine.

Stressing Apparatus.—This consists of a cast-steel hydraulic cylinder, with a ram 26 inches in diameter, and a 7-foot stroke. It is actuated by a water supply of 1,700 lbs. per square inch from an accumulator system. The ram passes through a U leather gland, coated with a sheath of electrically-deposited copper, to avoid the rusting to which steel rams are liable. The main ram pushes the movable bed of the machine, which slides on rollers. These rollers travel on planed and side-flanged cast-iron bed-plates which are bolted to the concrete foundations. The side flanges of the fixed bed-plates, guide and maintain the truly axial motion of the sliding bed.

General Description of the Working of the Testing Machine (see Fig. 7).— When the hydraulic ram H R advances from its hydraulic cylinder H C, the sliding bed S B carries with it the moving crosshead C_1, which is locked to it by means of four square locking bolts contained and guided

in the body of its crosshead. Parallel to the moving bed is a system of tie-rods T R, coupled near the cylinder H C by a cast-steel crosshead C_2, and at the other end of the machine by the crosshead C_3. These tie-rods T R are connected, as shown, to two weighing levers L_1, L_2 by the crosshead C_4.
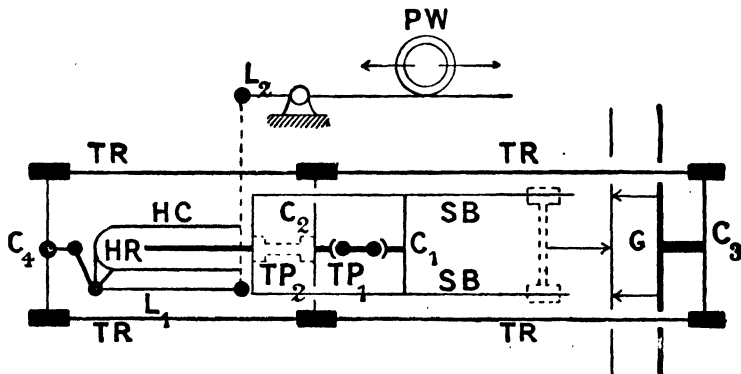
FIG. 7. — DIAGRAMMATIC LINE DRAWING OF BUCKTON & CO.'S 300-TON UNIVERSAL TESTING MACHINE FOR FULL-SIZED STRUCTURAL MEMBERS.

(*This sketch shows a test-piece* T P_1 *in tension, and in dotted lines a test-piece* T P_2 *in compression, also a bending test at* G.)

Equipment of the Machine for Different Kinds of Tests.*—

Tension Test.—The test-piece T P_1 is held between the moving crosshead C_1 and the fixed or weighing crosshead C_2. Each crosshead is made with a spherical seat, in which a spherical block is free to adjust itself, and contains the clips actually holding the specimens to be tested. For round or flat bars, serrated wedge clips, fitted in two semi-cylindrical backings, are inserted in the spherical block. By this means, even bars of unequal thickness, such as an angle-iron-flange, can be held. For testing specimens with solid heads, holders are inserted into the spherical blocks, which carry suitable dies made in halves. Tests on chains are made by means of four links. Keys pass through slot-holes in these links and hold the chains. Flat ropes are tested by means of two large drums, and are gripped at their extremity by clips. These same drums also lend themselves, by a slight modification, to testing round ropes.

Compression Test.—A compression test is effected by using the opposite sides of the crossheads C_1 and C_2 to those which serve for tension tests. In Fig. 7, a compression test upon a short test-piece T P_2 is shown between the crosshead C_2 and the end of the sliding bed S B, by means of compression plates 2 feet or 1 foot square respectively. To avoid any deviation of the crosshead C_2 from an axial line (which the

---

* This machine will test a strut 88 feet long by 3 feet 3 inches by 3 feet 3 inches ; or a chain 78 feet long, 4½ inches diameter, of iron ; or a beam 3 feet 3 inches broad, 6 feet 8 inches deep, by 20 feet between the supports. It will also test in single shear a bar 8 inches wide by 2½ inches thick.

crushing of a heterogeneous specimen might induce), this crosshead is guided in either direction by rollers mounted upon wedges, and the whole contained in strong vertical brackets well tied together.   Compression tests on long specimens are made between the crosshead C_1 and the deflection girder G, or the crosshead C_2, for the deflection beam G rests upon rollers and can be removed without difficulty.

Deflection Test.—For small specimens, this test can be carried out between the fixed crosshead C_2 and the end crosshead C_3 of the sliding bed ; to which are attached two small adjustable supports, as indicated by arrowheads.

Shearing and Punching Tests.—The punch and dies are inserted between the crosshead C_2 and the end of the sliding bed at C_1.   For shearing, there is attached to the crosshead C_2 a box in which the test-piece to be sheared is inserted.   The shear blade is guided by a roller inside the box, and by this means a simple shear is obtained exactly as in an ordinary shearing machine.

Autographic Diagrams.—By attaching recorder wires to the crossheads, autographic records can be taken of all the above tests.

A Convenient Method of Obtaining and Regulating the Pressure and Flow of Water to the Ram of a Testing Machine.*
—The method adopted by Prof. Kennedy in working the testing machine at University College, London, is shown by Fig. 8.

Here a is the delivery pipe to the testing machine; b an ordinary force pump, with its lever c; d an accumulator loaded to about $1\frac{1}{2}$ tons per square inch. The pump is worked by a Davey motor e, on the shaft of which an eccentric f drives the free end of a double bar link g. The inner end of this link swings on a lever h, which can be thrown in or out by a hand lever k. The stroke of the block driving the pump lever may be made to vary from zero to the full stroke of the eccentric, according to the position of the link. In this way, the stroke of the pump can be modified, or its action entirely stopped by the hand lever, without stopping the
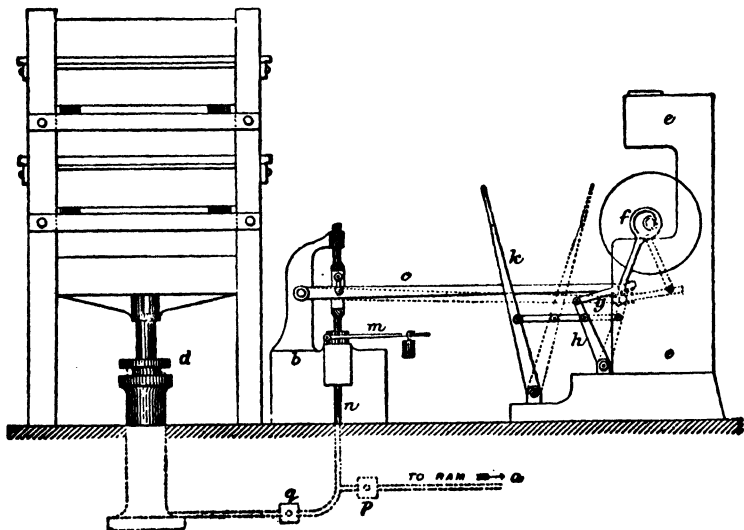
FIG. 8.—PROF. KENNEDY'S METHOD OF WORKING THE TESTING MACHINE.

engine. A relief valve m is fitted in the pump itself. And, n is the delivery pipe from the pump, with two regulating valves p and q. The valves allow the pump to pump either (1), to the accumulator, or (2), to the ram direct, or (3), the pump and accumulator to work simultaneously on the ram, or (4), the accumulator alone to work on the ram. For the taking of stress-strain diagrams and other delicate test-work, it is found, that it is better to use the accumulator alone, its speed being regulated by throttling the water passing through both the valves p and q. For ordinary work,

---

*.In the new *James Watt* Laboratory at Glasgow University, an accumulator with removable weights is used in connection with the large Buckton & Co.'s horizontal testing machine. The force pump is driven by belting from shafting, and is automatically set to work or stopped by the accumulator in a similar way to that in the figure in Lecture II., Vol. IV., "Hydraulics."

when a large number of specimens are to be tested and speed is an object, then the pump and the accumulator are employed simultaneously. The accumulator is pumped up the distance it has fallen during the replacement of the broken test-piece. The accumulator load is divided into four parts, any or all of which can be used simultaneously, or shelved, according to the required load.

The maximum of steadiness, as well as of convenience in working, will be found in some such system as that shown by Fig. 8, although it may not be necessary to use an accumulator for ordinary works' testing.

Proportions of Test-Bars and their Attachment to Tensile Tests.—The British Engineering Standard Committee have specified the following rules as to gauge length :—

(*a*) FLAT BARS.—Gauge length = 8″; parallel for 9″.

If the thickness is greater than ⅞ inch,
    Maximum width = 1½ inches.

If the thickness is between ⅜ and ⅞ inch,
    Maximum width = 2 inches.

If the thickness is less than ⅜ inch,
    Maximum width = 2½ inches.

(*b*) TURNED SECTIONS (diameter *d*).—

    Gauge length = 8 *d*; parallel for 9 *d*.

(*c*) TURNED SPECIMENS FROM FORGINGS. –

    Area, ¼ inch; gauge length = 2 inches.
     „ ½ „   „   = 3   „
     „ ¾ „   „   = 3½ „
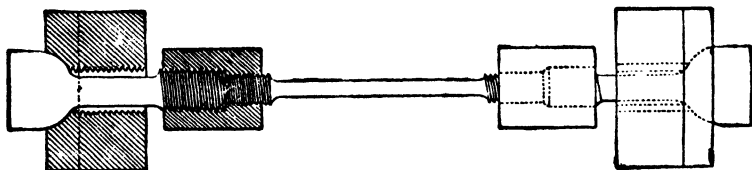
FIG. 9.—HOLDER FOR SECURING AND ADJUSTING THE TEST-PIECE
IN THE MACHINE.

The method of holding a test-piece so fair, that the pull will be symmetrically distributed about the axis of the test-piece, and also that fracture may not occur at or near the grips due to any inequality in the stress, always presents some difficulty, more especially when the material is of the rigid non-plastic type. The test-piece is sometimes made with enlarged ends on which screw-threads are lathe-out. Each end of the

test-piece is then screwed into a nut. The seat of the nut is so shaped as to form part of a sphere, thus forming a ball-and-socket joint at each end of the test-piece. Or, the enlarged ends terminate in shoulders inside of which two half rings are placed to form a collar, and the half rings are spherical-curved where they form the bearing with the shackle.

When testing plastic materials, such as the various kinds of wrought iron and mild steel, very little difficulty is experienced in obtaining a fair test of their ultimate strength, even with the very simplest appliances for holding the test-piece, because the plastic yielding which precedes the rupture of the material wipes out any inequality there may be in the distribution of the stress to begin with. Therefore, the trouble of screwing the ends of the test-piece or of forming shoulders may be dispensed with, and a much simpler method of attachment by wedge grips may be adopted. In this arrangement each end of the test-piece lies between two wedges of hard steel. The faces of these wedges where they press on the test-piece are rough, while the backs are made smooth and greased in order to make them slip easily along the tapered recess in the shackle. When the pull comes on the bar, the wedges are drawn along with it and press against the test-piece with sufficient force to cause the rough faces of the wedge to bite into the plastic surface of the test-piece and thus hold it securely. When testing flat plate strips the ends of the test-piece are cut a little wider than the main body of the piece, thereby giving an enlarged surface for the wedge to act on. No enlargement of the ends is required with round or square test-pieces, because the wedges, instead of being made plain, have a groove with roughened sides, so that each end of the test-piece is gripped at four places around its circumference. The tapered hole in which the wedges are placed is made from two half rings which are separately free to turn round in the shackle, thus permitting of its adoption to cases where the opposite sides of the strip are not perfectly parallel.
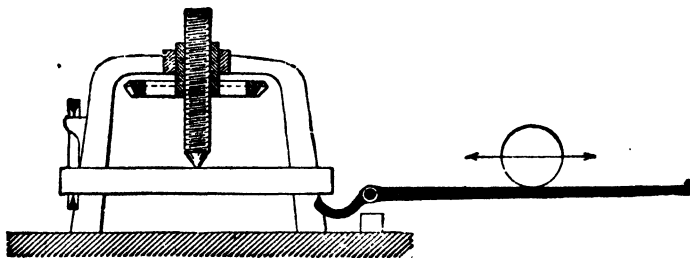
FIG. 10.—PROF. KENNEDY'S SMALL TESTING MACHINE FOR TRANSVERSE TESTS.

3. Transverse Tests.—Transverse and torsional tests usually require so much smaller loads than those used for tension and compression, that it is probably better to carry out the two former kinds of tests upon separate machines. Some arrangement for transverse tests is frequently fitted to large tensile testing machines, but Fig. 10 shows a simple, small, separate machine designed by Prof. Kennedy for laboratory transverse tests up to central loads of 4 tons and spans of 5 feet. It will be seen from the figure, that the load is applied by screw gearing to the centre of the span, and that its value is measured by a steelyard and poise-weight at one end,
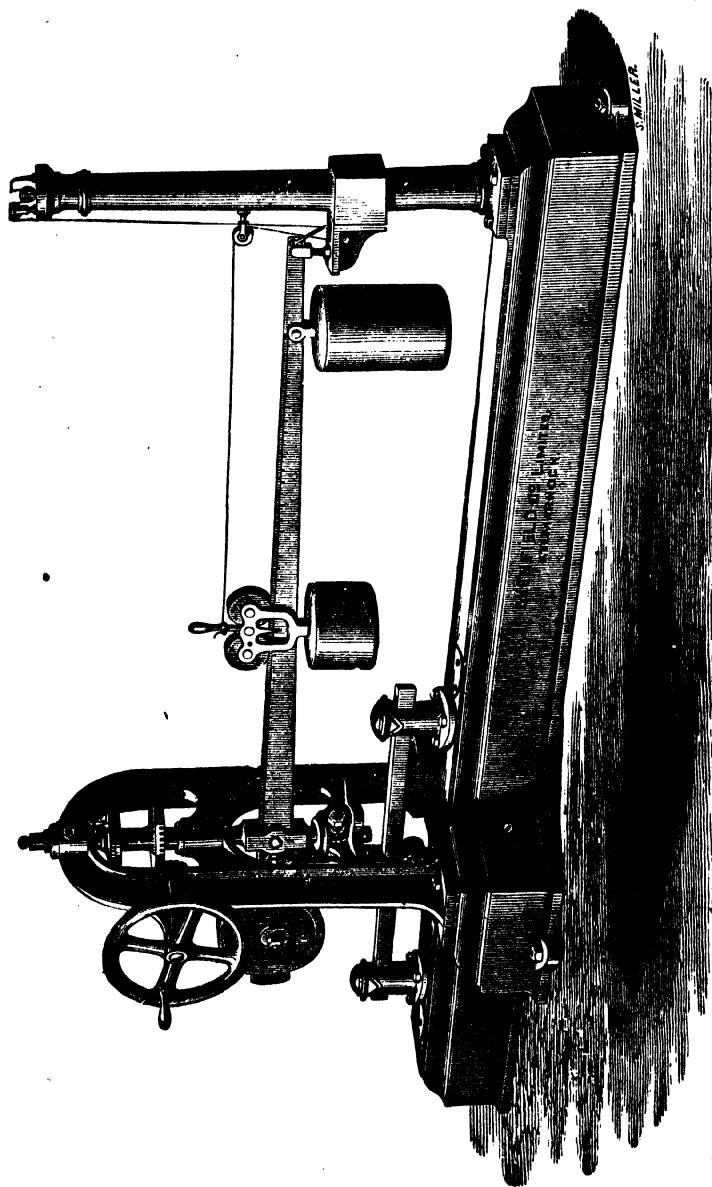
FIG. 11.—TRANSVERSE BAR TESTING MACHINE.   Made by Glenfield & Kennedy, Limited, Kilmarnock.

Transverse Bar Testing Machine. — Engineers and architects usually specify that cast-iron beams, pipes, columns, &c., shall be made of a certain brand of iron, and that standard specimens cast from the same ladle shall withstand certain deflections and bending stresses. It is, therefore, of great importance that cast-iron founders should possess a simple and reliable machine whereby such tests may be quickly and accurately made.

The accompanying illustration shows one form of such a machine for testing the ordinary standard sizes of specimens of cast-iron bars—viz., 2 inches × 1 inch, or 1 inch square, with centres 3 or 4 feet apart. After introducing the test-bar to the position shown by its supports on the sole plate of the machine, the hand-wheel is turned until the stress just makes the test-bar taut on the upper knife-edges of the supports. Then, the "deflection index" is set at zero. Turning the hand-wheel still further causes the right-hand end of the weigh-beam lever to rise. This action releases the small catch, fixed to the end of the lever, from the ratchet wheel, which allows a weight in the cataract cylinder placed in the right-hand end pillar to pull the travelling weight towards the right hand along the top of the graduated lever. Should the travelling weight and its index pointer arrive at the end of the lever before the specimen is broken, the travelling weight is brought back to zero and an additional weight is then added to the hanging rod, and the above process is repeated. When the bar breaks, an index, not shown on the figure, is connected to the machine when in use, which shows the maximum deflection of the test-piece to the third decimal place of an inch, whilst the stress in lbs. required to break the specimen is shown by the position of the index on the movable weight opposite the scale figure on the weigh-beam bar together with the weights on the vertical rod.

4. Cross Bending, Folding, or Doubling Tests—Cold, Hot, or after Annealing.—The mechanical tests of the cast and of the annealed steels, illustrated above and below the photo-micrograph plate of Fig. 1 in this lecture, consisted of simple *cross bending* over a radius ⅜ inch in the case of the test-piece from the fresh casting, and of doubling after careful annealing over a rod ¾ inch in diameter.
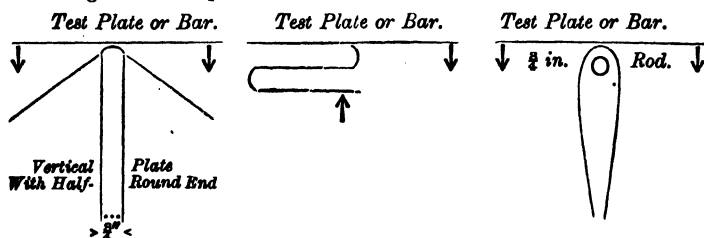
FIG. 12.—CROSS BENDING.   FOLDING.   DOUBLING.

Fig. 12 shows these three different tests. They may be carried out cold as the test-piece comes from the foundry or the forge; or, after any amount of specified annealing, according to circumstances.

In the following set of rules by the British Admiralty, and in the suggestions by the French Commission on Testing, we see such tests are applied. Sometimes, they produce a very severe stress in the material, and, if properly carried out and fulfilled, they are most certainly a surety that the metal possesses ductility, pliability, and tenacity.

Admiralty Rules for Testing Materials for Machinery.—

1. *Steel Castings for Machinery.*—Steel castings for the machinery are to satisfy the following conditions :—Tensile strength, not less than 28 tons per square inch, with an extension of 2 inches of length of at least 23 per cent. Bars of the same metal, 1 inch square, should be capable of bending cold, without fracture, over a radius not greater than 1⅝ inches, through an angle depending on the ultimate tensile strength, this angle to be not less than 90° at 28 tons ultimate strength, and not less than 60° at 35 tons ultimate strength, and in proportion for strengths between these limits. For intricate thin castings the extension in 2 inches of length is to be at least 10 per cent., and the bending angle is to be not less than 20° at 28 tons ultimate strength, and 15° at 35 tons ultimate strength, and in proportion for strengths between these limits. Test-pieces are to be taken from each casting. All steel castings are also to satisfactorily stand a falling test, the articles being dropped from a height of 12 feet (or as may be approved) on a hard macadamised road or a floor of equivalent hardness.

It is to be distinctly understood that contractions or defects in steel castings are not to be made good by patching, burning, or by electric welding, without the sanction of the Admiralty overseers.

2. *Steel Forgings for Machinery.*—All steel forgings are to satisfy the following conditions :—Ultimate tensile strength, not less than 28 tons per square inch, with an extension in 2 inches of length of at least 30 per cent. Bars of the same metal, 1 inch square, should be capable of being bent cold, without fracture, through an angle of 180° over a radius not greater than ¼ inch. Test-pieces are to be taken from each forging. Crank and propeller shafts are to have test-pieces taken from each end, and the ultimate tensile strength of the material of these shafts must not exceed 32 tons per square inch.

For all important forgings, such as crank and propeller shafts, connecting- and piston-rods, the forgings are to be gradually and uniformly forged from solid ingots, from which at least 30 per cent. of the top end of the ingot has been removed before forging, and at least 3 per cent. of the total weight of the ingot from the bottom end after forging. The sectional area of the body of the finished forging is to be not more than ⅛ the original sectional area of the ingot.

3. *Cast Iron.*—Test-pieces to be taken from such castings as may be considered necessary by the inspecting officer. The minimum tensile strength to be 9 tons per square inch, taken on a length of not less than 2 inches. The transverse breaking load for a bar 1 inch square, loaded at the middle between supports 1 foot apart, is not to be less than 2,000 lbs.

4. *Gun-metal, Naval Brass, and White Metal.*—The gun-metal used for all castings throughout the whole of the work supplied by the contractors is, unless otherwise specified, to contain not less than 8 per cent. of tin, and not more than 5 per cent. of zinc, the remainder to be of approved quality copper. The exact proportion of tin above 8 per cent. being arranged as may be required, depending on the use for which the gun-metal is intended. The ultimate tensile strength of gun-metal is to be not less than 14 tons per square inch, with an extension in 2 inches of length of at least 7½ per cent. The composition of any naval brass used is to be :—

2

Copper, 62 per cent.; zinc, 37 per cent.; and tin, 1 per cent. All naval brass bars are to be cleaned and straightened. They are to be capable of (1) being hammered hot to a fine point, (2) being bent cold through an angle of 75° over a radius equal to the diameter or thickness of the bars. The ultimate tensile strength of naval brass bars ⅞ inch diameter and under is not to be less than 26 tons per square inch, and for round bars above ⅞ inch diameter, and square bars not less than 22 tons per square inch, whether turned down in the middle or not. The extension in 2 or 4 inches of length is to be at least 10 per cent. Breaks within less than ½ inch of the grip are not to count. Cuttings from the propellers and other important gun-metal castings and naval brass work will be sent to Portsmouth dockyard for analysis. The white metal used for bearing surfaces is to contain at least 85 per cent. of tin, not less than 8 per cent. of antimony, and about 5 per cent. of copper; zinc or lead should not be used. The brasses are to be carefully tinned before filling with white metal.

Castings and forgings. The gun-metal castings are to be perfectly sound, clean, and free from blowholes. The steel castings are required to be clean, sound, and to be out of twist, and as free as possible from blowholes; the steel forgings are required to be quite sound, clean, and free from all flaws. All castings and forgings must admit of being machined, planed, and bored to the required dimensions; and no piecing, patching, bushing, stopping, or lining, will be permitted, nor will any manufactures in which these conditions have been infringed be accepted. In cases of doubt as to the suitability of castings or other materials, for the purpose intended, early reference should be made to the inspecting officer to avoid subsequent delay by the rejection of such parts after delivery. The whole of the steel plates, angles, and rivets used in the construction of any part of the work supplied is to be manufactured by the Siemens-Martin process. All castings are to be of steel or gun-metal, except where otherwise specified; and all forgings, plates, bolts, &c., are to be of steel. No steel is to be toughened without sanction from the Admiralty.

Copper. 5. *Copper for Pipes.*—Strips cut from the steam and other pipes, either longitudinally or transversely, are to have an ultimate tensile strength of not less than 13 tons per square inch when annealed in water, with an elongation in length of 2 inches or 4 inches of not less than 35 and 30 per cent. respectively. Such strips are **Hammer test.** also to stand bending through 180° cold until the two sides meet, and of hammering to a fine edge without cracking.

6. *Treatment of Mild Steel.*—All plates and furnaces for boilers **Scale to be removed.** and steam pipes are to be treated as follows, with a view of removing the black oxide or scale formed during manufacture:—The plates and furnaces, previous to their being taken in hand for working, are to stand for not less than 8 hours in a liquid consisting of 19 parts of water and 1 of hydrochloric acid. The plates should be placed in the bath on edge, and not laid flat. When the plates and furnaces are removed from the dilute acid both the surfaces are to be well brushed and washed to remove any scale which may still adhere to them. They should then be placed in another similar bath filled and kept well supplied with fresh water, or be thoroughly washed with a hose, as may be found necessary. The plates on removal from the fresh water should be placed on edge to dry. This treat-

ment is to be carried out on the premises where the boilers and
pipes are made.

All plates or bars which can be bent *cold* are to be so treated ; Cold
and if the whole length cannot be bent cold, heating is to be had bending.
recourse to over as little length as possible.

The front and end plates of the boilers, and also all other plates, Hy-
including those for the combustion chambers *are to be flanged by* draulic
hydraulic pressure, and in as few heats as possible. flanging.

In cases where plates or bars have to be heated, the greatest care Danger-
should be taken to prevent any work being done upon the material ous tem-
after it has fallen to the dangerous limit of temperature known as a to be
"blue heat"—say from 600° to 400° Fahrenheit. Should this limit avoided.
be reached during working, the plates or bars should be reheated.

Plates or bars which have been worked while hot are to be subse- Anneal-
quently annealed simultaneously over the whole of each plate or bar. ing.

In cases where any bar or plate shows signs of failure or fracture Failure
in working it is to be rejected. Any doubtful cases are to be of metal.
referred to the Admiralty.

French Methods of Testing Materials of Construction.*—

Vol. I., in addition to the constitution of the commission, contains the
reports of the two sections. That of Section A is divided into four parts
as follows :—

Part I. Physical tests :—(*a*) Microscopic and other examination of the
exterior appearance and fractures, and tests by resonance ; (*b*) determina-
tion of density and thermal and electrical conductivity ; (*c*) determination
of the critical temperature and variations of conductivity with temper-
ature ; (*d*) study of temper.

Part II. Chemical tests :—(*a*) Analysis ; (*b*) corrosion and means of
protection.

Part III. Mechanical tests :—(*a*) General remarks ; (*b*) on samples ;
(*c*) the influence of heat ; (*d*) the influence of duration of test ; (*e*) testing
machines ; (*f*) dimensions ; (*g*) mechanical terminology ; and (*h*) technical
terminology.

Part IV. Methods of mechanical testing :—(*a*) Gradually applied tests,
tensile, compressive, bending, folding and doubling, torsional, shearing,
and punching ; (*b*) tests by blows, bending, indentation, and perforation ;
(*c*) study of hardness and brittleness ; (*d*) hot and cold working, ham-
mering, &c. ; (*e*) special tests of wires, ropes, chains, rivets, pipes and
tubes, and hydraulic pressure tests.

The report of Section B, which deals only with cements, limes, and
kindred materials, is divided into the following parts :—

Part I. General considerations, choice of tests, normal tests, &c.

Part II. Tests of cement, fineness, specific weight, apparent density,
analysis, homogeneity, making of sample briquettes and mortars, setting,
tensile, compressive, and bending tests ; stability, efficiency, porosity,
permeability, solubility in sea water, and adhesiveness.

Part III. Tests of hydraulic and fat limes.

Part IV. Tests of puzzolanas, &c. Part V. Sand for use in mortar.

Part VI. Plasters. Part VII. General conclusions.

---

*The original report of the proceedings of the French Commission on
"Methods of Testing Materials of Construction" will be found in the
library of the *Inst. C. E.*, dated Paris, 1894.

LECTURE V.—QUESTIONS.

1. State a few of the chief cases in the history of testing machines.

2. What are the advantages to be derived by young engineers from a laboratory course on testing the strength and elasticity of materials, and why?

3. Why is it advisable to obtain not only accurate mechanical tests, but also chemical analysis and photo-micrographs of specimens of the materials to be used before designing an engineering structure and during their manufacture?

4. State the different mechanical tests which you could apply to, say, steel bars or plates. Explain by aid of an example why impact tests are important.

5. Suppose that you were called upon to investigate the case of a broken connecting-rod from a high-speed reciprocating engine, state the different things and information you would require before drawing up your report.

6. Sketch and describe Prof. Barr's wire-testing machine, state how you would use it and what information you would get from your results.

7. Give free-hand line diagram sketches of three leading types of testing machines.

8. Sketch and describe a good 30- or 50-ton single-lever vertical testing machine. State how it works and how you would obtain reliable results from it.

9. Give a line diagram of a 300-ton universal testing machine, and state how it works for tension, compression, and bending stresses.

10. Sketch and explain any convenient method of obtaining and regulating the pressure and flow of water to the ram of a testing machine.

11. In making tensile tests of bars and plates what proportions would you adopt, and why? Sketch and describe a suitable holder for round test-pieces. When testing round or square bars of wrought iron or mild steel how would you shape and grip them?

12. What is the simplest form of machine that you know for carrying out simple cross-bending or transverse tests?

13. Sketch and describe the construction and action of an accurate transverse bar testing machine for measuring the deflections and bending stresses obtained with standard sized specimens of cast iron.

14. Explain, by aid of sketches, how you would carry out cross bending, folding, and doubling tests upon a steel bar or copper plate. Give instances where one or more of these tests are required by the Admiralty or the French Government, and mention the circumstances attending a proper fulfilment of the test.

LECTURE V.—A.M.INST.C.E. EXAM. QUESTIONS.

1. Describe the method of conducting a tension test of a bar of mild steel. State what precautions should be taken in preparing the test bar and what measurements should be made.

2. Describe and sketch one form of testing machine. State any special conditions you would require in such a machine for testing to 50 tons load. State how or to what extent its accuracy can be ascertained by trial.

3. Describe the methods of holding tension specimens, and mention any special advantages or defects of any of these.

4. Give a specification of tests of mild steel bars and plates for bridge-work.

5. Having ascertained that a certain mixture of cast iron possesses an ultimate tensile strength of 9 tons per square inch, calculate the breaking weight of a standard bar 2 inches by 1 inch, placed upon supports 3 feet apart and loaded in the middle. In using for this purpose the ordinary theory of transverse flexure, add any comments that you may think necessary in regard to the result so obtained.

6. Mention some practical tests by which we can estimate or measure the ductility of metals, and some of the reasons which make it important to obtain a test of this property.

7. Describe with the help of sketches the general arrangement of any testing machine you are familiar with, suitable for tensile and compressive tests. Explain how the machine may be tested for accuracy.

8. Explain how a cube of cast iron can be tested to destruction by compression in a testing machine; how is it likely to fracture? Give your reasons.

9. Sketch and describe any arrangement in which a weight can be moved along a beam which oscillates about a fixed fulcrum—such as the beam in a testing machine—without exercising any constraint on the oscillatory motion of the beam.

10. Specimens of (a) cast iron and (b) mild open-hearth steel are to be tested in direct tension. Sketch the forms of test-piece and of holding shackle that would be suitable in each case, and describe the kind of fracture that may be expected to take place.

11. In testing specimens of mild steel, with a given length between gauge-marks, how would the ultimate elongation be affected by the diameter of the test-piece.

12. Describe, with sketches, and explain the working of any form of autographic apparatus for producing a stress-strain diagram during the testing of a specimen to destruction in tension.

LECTURE VI.

STRENGTH AND ELASTICITY OF MATERIALS.
TESTING AND TESTING MACHINES—*Continued.*

CONTENTS.—Shearing, Planing, Boring, Punching, Drifting, and Riveting, Hammering to a Point or Edge—Fatigue of Metals—Historical Notes—Wöhler's Testing Machines, with an Example—General and Particular Deductions from Wöhler's Experiments—Gerber's Formula—Elastic Limits—Heating due to Stress—Rest and Heating Relieve Stress—Microscopic Examination of the Inter-Molecular Crystalline Action in Metals due to Fatigue—Prof. Ewing's Conclusions—Prof. Reynolds and Dr. J. H. Smith's Method of Testing Metals by Tensile and Compressive Reversals of the Mean Stress—Determination of the Tensile, Compressive, and Range of the Reversal Stresses, with Formulæ and Results on Mild Steels—Impact Tests—Seaton and Jude's Impact Testing Machine—Advantages of the Impact Test—Cumulative Effects of Small and Medium Shocks—Effect of Allowing a Factor of Safety for both Fatigue and for Impact—Drop or Falling Tests—Torsional Tests—Torsion of Rods and Wires—Hardness, Temper and Brittleness Tests—Methods of Determining the Hardness of a Metal—Workshop Test—Calvert and Johnson's Test for Alloys—Prof. Unwin's Test—Comparison between Unwin's Indentation Test and the Scratch Method—Brinell Hardness Test—Caledonian Railway Company's (1908) Specification for Steel Rails, 90 lbs. per Yard Section—Questions.

5. Shearing, Planing, Boring, Punching, Drifting, and Riveting, Hammering to a Point or Edge.—If this had been a lecture upon these headings in the sense of "Workshop Appliances and Methods," then many well-illustrated and instructive pages might be written about them. But here, in this instance, the attention of the student has simply to be drawn to the fact, that under certain circumstances, and in certain specifications, one or more of these methods are used as tests for iron, steel, and certain other metals, in addition to one or more of the special tests detailed in the previous, this, and the following lectures.

When Mr. H. E. Yarrow, of the well-known torpedo-boat building firm at Poplar, London, delivered his address to the Junior Institution of Engineers on December 2nd, 1904,* he pointed out, that one of the most important considerations in the design of torpedo vessels was, how to realise (as far as possible) the maximum of strength with the minimum of weight, so that their hulls might have uniform elasticity throughout their length. The importance of realising this fact will be understood when it is remembered, that in a rough sea such vessels are subjected to severe bending

forces many times per minute. It, therefore, lies in the capability of the
yielding qualities of such fast vessels to the blows of the sea to partially
and gradually absorb the energy of the waves — just like the action
on a watch spring—rather than to rigidly oppose these forces, that
their success depends.   Any *sudden change* in the elasticity of the
plates, due to local hardening or stiffening, will sooner or later tend to
fracture them where the rigidity begins and ends. You may say—but
what has all this got to do with the above-mentioned headings? Well,
we shall see.

Shearing versus *Planing, and Punching* versus *Boring* —The operations
of shearing and of punching ordinary iron or steel plates for ships, bridges,*
or boilers are certainly not mild, clean, kind, cutting actions.   They con-
sist largely of tearing and detrusion, for most of the work is done at the
commencement of the pressure of the shears or the punch, or just imme-
diately after the first yielding of the metal.

The processes of shearing and of punching require far greater immediate
forces to accomplish their object than planing or boring by keen well-
shaped tools. Consequently, the plates are much more stressed, strained,
and hardened or damaged near the shear line and around the punched
holes than they would be if they were well planed and bored. Of course, it
depends largely upon the class of work to be done, and the kind of material
used, whether shearing or planing and punching or boring should be
employed.   For the commoner kinds of cheap "cargo-wallers" or tramp
vessels, as well as for ordinary steel roofs and bridges, shearing and
punching of their plates and angle-irons may be good enough ; but, for
high-class steam boilers, warships of all kinds, and the best engineering
structures which have to resist severe pressures, alternating stresses and
shocks, planing and boring is now rigidly insisted upon.   And, the plates,
or samples of them, should be tested for their respective adaptability to
undergo the agreed upon constructive operations.

Punching and Drifting.—When holes are to be punched in plates which
have afterwards to be riveted together, the positions of the holes are
generally marked off on each of the two plates from the same wooden
template.   Then, however accurately pitched the holes in the wooden
template may be, and however great the care in marking them off upon
each of the two plates, the holes seldom come in quite fair and square
when the plates are set up for riveting. Consequently, in order to bring the
holes into line, severe drifting with a tapered steel tool or drift and a power-
fully wielded forehammer is generally resorted to by the riveter before he
can introduce the rivet.   Rimering or broaching out the holes by a tapered
cutting tool would be a much kinder action, and would relieve some of
the stresses in the metal, but then the holes would often be skew or
slanting, and of different sizes.   In any case, the double operation of
punching and drifting is much more likely to damage the plates, and
tend to their afterwards cracking and splitting between the holes, than
the following method, which is now being adopted in all the best Admiralty
and other work where the riveting of plates have to be effected :—

(1) One plate is drilled and, if need be, counter-sunk on its outer side.

(2) The other plate is then set up in position and both are clamped
firmly together.

(3) If the plates form part of a structure such as a ship or a bridge where
an ordinary drilling machine cannot be got to act, then pneumatic or
electric drills are set to work to bore out the holes in the unpierced
plate from the side and by the aid of the guidance of the holes in the
previously drilled plate, which thus acts as "jigs" or guides.

* Bridges are all *drilled* and *planed*

In this way, the work may be done with very great accuracy and with an entire absence of the necessity for drifting, as well as pleasure to the workmen, thus tending to his evenness of temper and absence of strong language, which is usually met with when punched holes require much drifting, accompanied by a corresponding punishment to the plates. Whenever several of the first set of plates have the same size and pitch of holes, they may be clamped together and multiple-drilled all at once. Thus, a saving in the expense as well as infinitely better treatment may be effected, so that the drilling of holes in plates which have to be afterwards riveted together will not cost very much more than punching them.

Hammering to a Point or Edge.—Punching or drifting as a mere test is sometimes resorted to, as well as that of hammering, cold or hot, to a point or edge.*

It is sometimes found, that in certain kinds of steel plates the metal is not homogeneous. If this should be suspected, then a sample plate may be taken and punched here and there, when it will soon be noticed whether the punch passes through all the holes with the desired uniform facility or not, and whether the parts surrounding the holes have suffered in any way.

Also, drifting tests are sometimes resorted to as a means of detecting the ductility or fitness of metals to withstand cold expanding, squeezing, riveting, or caulking stresses, such as in the case of boiler tubes and rivets, &c. The old familiar "Admiralty Test," of hammering down to an edge or point whilst cold or hot certain specimens taken from a plate or a bar is still quite common, as may be seen by inspecting the Admiralty rules and the French methods of testing materials of construction, given at the end of the previous lecture; or, in fact, most of the rules of the several inspecting Boards referred to in the footnote on page 652.

These and such like tests depend to a certain extent upon the skill of the workman who performs them, and a man either "makes a spoon or spoils a horn," if he wishes to do so, when the metal is not specially well adapted for the test in question. In this way, then, the tests under this heading have to be performed by an unbiased and skilful hand, and are thus unlike the previously mentioned mechanical machine tests.

6. Fatigue of Metals.

This is a most important part of the subject *Strength and Elasticity of Materials.* It has received special attention from a few eminent engineers, whose laborious and valuable researches are recorded by them in the *Proceedings* of the learned Societies or Institutions to which they belonged. But, as these are unobtainable by students, except through one or other of the chief engineering or university libraries, a short history will not be out of place.

Historical Notes.—In 1860, Sir William Fairbairn, LL.D., F.R.S., a Scotchman, carried out a series of the first recorded experiments on *Repeated Stresses* upon a riveted wrought-iron girder; and in 1864 he read a paper on his experiments before the Royal Society. These results seem to have agreed with those arrived at quite independently by Wöhler.

From 1860 to 1870 Herr A. Wöhler, formerly Locomotive Superintendent of the Royal Lower Silesian Railway, was engaged by the Prussian Ministry of Commerce to conduct a series of experiments on the capabilities of iron and steel to withstand repeated alternating stresses.

---

* See *Proc. Inst. C.E.*, vol. xcii., for a paper on "The Use of Testing of Open-Hearth Steel for Boilermaking," by Hamilton Goodall.

His laborious and most valuable researches into the "Fatigue of Wrought Iron and Steel" are published in *Zeitschrift für Bauwesen*, Berlin. A very full abstract of his report, with figures of the different machines designed and used by him, appeared in *Engineering*, vol. ix., 1871. A shorter article by Arthur N. Kemp on the "Fatigue of Metals" appeared in the September number of the *Engineering Review* for 1904, wherein some of Wöhler's machines and experiments are given, as well as results by other engineers.

In 1873, Launhardt's experiments are given in *Zeitschrift des Architecten und Ingenieur-Vereins*, Hanover.

In 1874, Spangenberg repeated Wöhler's experiments with the very machines devised by the latter, and obtained similar results, which are also published in *Zeitschrift für Bauwesen*.

In 1874, Gerber published in *Zeitschrift für Bankunde*, Munchen, a formula representing the best results of Wöhler's experiments. It is certainly simpler and easier of application than Launhardt's or Weyranch's formulæ, and is the same or similar to that used by Prof. Unwin (see a following page).

In 1879, Lippold published his views in *Organ für die Fortschritte des Eisenbahnwesens*, Wiesbaden.

In 1881, Prof. Mohr wrote on this subject to *Der Civil Ingenieur*, Leipzig.

In 1886, Sir Benjamin Baker, M.Inst.C.E., read a paper before the *American Society of Mechanical Engineers* on the results of his experiments upon iron and steel, which were obtained by using a similar rotating machine to one of those designed and used by Wöhler. Here, a horizontal spindle rotated 50 to 60 times a minute, having chucks beyond its two midbearings gripping test bars, whose further or outer extremities were hooked on to vertical tension springs. Hence, during each revolution, each test-piece was subjected by its spring to alternate tension and compression. In one case these were repeated over 14×10^6 times before fracture of the test-piece occurred.

In 1886, Bauschinger published in *Mittheilungen aus dem Mech. Techn. Laboratorium in München* his important communication on the "Variation of the Elastic Limits of Materials."

In 1899, Prof. J. A. Ewing, F.R.S., and Walter Rosenhain, in their Bakerian lectures on the "Crystalline Structure of Metals," and again in the *Philosophical Transactions of the Royal Society* (see Series A, vols. 193 and 195), give some interesting and useful information bearing upon this subject.

In 1902, Prof. Osborne Reynolds, F.R.S., and Dr. J. H. Smith, M.Sc., read their paper on a "Throw-testing Machine for Reversals of Mean Stress" before the Royal Society (see Series A, vol. 199 of *Phil. Trans. Roy. Soc.*). I am indebted to the kind permission of Dr. J. H. Smith for leave to make abstracts of this paper.

In 1905 Dr. Stanton and Mr. Bairstow read an important paper giving the results obtained at the National Physical Laboratory (*Proc. Inst., C.E.*, vol. clxvi.).

In 1911 Messrs. Eden, Rose, and Cunningham (*Proc. Inst., M.E.*), published a paper giving the results of experiments upon rotating beams with uniform bending moment.

Wöhler's Testing Machines, with an Example.—By means of ingeniously contrived machines, he submitted test-pieces to—(1) Direct pull, alternated with complete or partial relaxation from the pull. (2) Repeated bending in one direction. (3) Bending in opposite directions. (4) Twisting the test-piece towards one side ⌒⌄ only ; and, (5) towards opposite sides ⌁⌄

Unfortunately, Wöhler's experiments, although extensive, were not carried out on a sufficient variety of materials to be of universal use since his experiments were restricted to iron and steel ; hence the results of his tests must not be generalised too freely. He found, that wrought iron, having an ultimate tensile strength of 19·5 tons per square inch, would stand an indefinite number of alterations of stress, provided the limits did not exceed the following :—

Kind of Stress.	Stress in Tons per Square Inch.
Alternating load, or from a pull to a push, . .	+ 7 to − 7
Load varying between zero and maximum, or from a certain pull to no stress, ...	+13 ,, 0
Load varying between limits, or from a certain pull to a less pull, ...	+19 ,, +10½

From these results it appears, that the relative values of the maximum loads for iron under these three conditions are approximately 1 : 2 : 3. Moreover, he found similar results in the case of certain specimens of steel.

General and Particular Deductions from Wöhler's Experiments.—(1) That wrought iron and steel are ruptured by stresses *much below* their statical breaking stress, if such stress *be repeated a sufficient number of times.*

(2) That within certain limits the *range of stress*, and not the maximum stress, determines the number of reversals for rupture.

(3) That as the *range of stress* is diminished, the number of repetitions for rupture increases.

(4) That there is a *limiting range of stress* for which the number of repetitions of stress for rupture becomes infinite.

(5) That this *limiting range of stress* diminishes as the maximum stress increases, which was clearly indicated by the results under the previous heading.

Gerber's Formula.—As previously mentioned in the historical notes, Gerber founded a formula upon Wöhler's experiments. He did so by simply plotting the *ranges of stress* as ordinates and the corresponding minimum stresses as abscissæ, when he found, that their points of inter-section, if joined, formed a parabola. Consequently, the formula to fit in with this curve is known as—

GERBER'S PARABOLIC EQUATION

$$f(\text{max.}) = \tfrac{1}{2}\Delta + \sqrt{(f^2 - c\,\Delta f)}.$$

Where $f(\text{max.})$ = the maximum stress,
  ,, $f(\text{min.})$ = the minimum stress,
  ,,   . f = the statical breaking stress,
  ,, Δ = the range of stress = $f(\text{max.}) \pm f(\text{min.})$,
And c = a constant.

For a statical load, $\Delta = 0$, and $f(\text{max.}) = f$.
For a load varying between 0 and a maximum—

$$\Delta = f(\text{max.}).$$

And approx., when $c = 1\cdot5$, then $f(\text{max.}) = \tfrac{2}{3}f$.
For an $\sim$ load varying between $f(\text{max.}) +$ and $f(\text{min.}) -$

$$\Delta = 2f(\text{max.}).$$

And, when $c = 1\cdot5$, then $f(\text{max.}) = \tfrac{1}{3}f$.

Hence, the relative values of the above maximum loads, which are —

	Alternating	:	Variable	:	Statical,
Are as,	$\tfrac{1}{3}$	:	$\tfrac{2}{3}$	:	$\tfrac{2}{3}$,
Or as,	1	:	2	:	3,

just as Wöhler found them to be, by his testing machines, for the wrought iron referred to in a previous paragraph. Of course, the constant c has various values for different materials, and, whilst it may be about $1\cdot5$ for a certain kind of wrought iron, it will rise as high as $2\cdot5$ for the finest spring steel.

EXAMPLE I.—Two steel bars, having a static breaking load of 28 tons per square inch, are stressed in tension, the one from 4 to 5 tons per square inch and the other from 1 to 5 tons per square inch. Find the breaking strengths for the respective methods of loading.

The stress variation must first be put in terms of maximum static stress, and this again in terms of the new breaking stress $f(\text{max.})$, thus :—

$$\Delta = \frac{\text{Highest stress} - \text{lowest stress}}{\text{Highest stress}} \times f(\text{max.}).$$

In the first case, where the ultimate static stress = 30 tons per sq. inch,

$$\Delta = \frac{5-4}{5}f(\text{max.}) = \frac{1}{5}f(\text{max.}).$$

Now, taking Gerber's formula and substituting the values for the symbols.

Where f = Original breaking stress in tons per square inch = 28.
  ,, Δ = Stress variation in terms of $f(\text{max.})$ in tons per sq. inch.
  ,, c = A constant deduced from experiments = 2 for hard steel.
  ,, $f(\text{max.})$ = The new max. breaking stress in tons per sq. inch.

We get, $f(\text{max.}) = \dfrac{\Delta}{2} + \sqrt{(f^2 - c\,\Delta f)}$

$$,, \quad = \frac{1}{10}f(\text{max.}) + \sqrt{(28)^2 - \left(2 \times \frac{1}{5}f(\text{max.}) \times 28\right)}.$$

$$\frac{9}{10}f(\text{max.}) = \sqrt{784 - 11\cdot2\,f(\text{max.})}.$$

Simplifying, squaring, and solving this quadratic equation, we find—

For *First Case*—

$$f\text{(max.)} = 22\cdot5 \text{ tons per square inch.}$$

In the second case, where the ultimate static stress = 28 tons per sq. inch,

$$\Delta = \frac{5-1}{5}f\text{(max.)} = \frac{4}{5}f\text{(max.)}.$$

Therefore, substituting the values in the formula, we obtain—

$$f\text{(max.)} = \frac{\Delta}{2} + \sqrt{(f^2 - c\,\Delta f)}$$

$$\text{,,} \quad = \frac{2}{5}f\text{(max.)} + \sqrt{(28)^2 - \left(2 \times \frac{4}{5}f\text{(max.)} \times 28\right)}.$$

$$\frac{3}{5}f\text{(max.)} = \sqrt{784 - 44\cdot8\,f\text{(max.)}}.$$

Simplifying, squaring, and solving the quadratic as in the first case, we find—

For *Second Case*—

$$f\text{(max.)} = 14\cdot6 \text{ tons per square inch.}$$

Elastic Limits.—Bauschinger showed, in his 1886 paper, that when the elastic limit in tension is raised, the elastic limit in compression is lowered. Also, by subjecting a material to a few alternations of equal stresses, the elastic limits tend towards fixed positions, which positions he called the *natural* elastic limits. Further, he explained the results which he obtained, when subjecting a metal to repeated stresses, by showing that the limiting range of stress coincided with the difference of the two natural elastic limits.

Heating due to Stress.—All have noticed, that when bars of metal are subjected to various stresses they become heated due to the work done upon them. Hence perfect elasticity does not exist in such materials.

Rest and Heating Relieve the Molecular Stress.—It has been noticed, that when a metal has been fatigued, due to varying stresses, a period of rest or of annealing will apparently restore its elasticity. Profs. Ewing and Muir of Cambridge are reported to have found that a few minutes' immersion of a fatigued bar in hot water effects such a recovery; thus showing that high temperatures are not necessary under certain conditions for annealing or for recuperation.

Microscopic Examination of the Inter-Molecular Crystalline Action in Metals due to Fatigue.—Prof. Ewing and Mr. J. C. W. Humphrey,[*] in some recent experiments, applied the microscope to watch the process by which iron breaks down and becomes fatigued under repeated reversals of stress. The tests which they carried out were on

---

[*] See *Proc. Roy. Soc.*, 1903, paper on "The Fracture of Metals under repeated Alterations of Stress," by Prof. Ewing and J. C. W. Humphrey.

Swedish iron of high quality, having a statical breaking strength of 23·6 tons per square inch on the original area, an extension of 26 per cent. in a length of 8 inches, and a contraction of 61 per cent. in the area.

The test-bars were in the form of rectangular rods measuring ·3 inch by ·1 inch, and were first prepared by annealing in a muffle furnace, being enclosed in a tube of lime, and the whole kept at a dull-red heat for two hours. One face of each rod was polished and etched prior to its being subjected to alternating stress actions, so that under test the polished surface was alternately stretched and compressed. The machine used was similar in form to that of Wöhler's, and ran at 400 revs. per minute.

Prof. Ewing's Conclusions.—" Whatever the selective action of the stress is due to, the experiments demonstrate, that in repeated reversals of stress certain crystals are attacked and yield by slipping, as in other cases of non-elastic strain. Then, as the reversals proceed, the surfaces upon which slipping has occurred continue to be surfaces of weakness. The parts of the crystal lying on the two sides of each surface continue to slide back and forward over one another. The effect of this repeated sliding or grinding is seen at the polished surface of the specimen by the production of a burr or rough and jagged irregular edge, broadening the slip band and suggesting the accumulation of *debris*. Within the crystal this repeated grinding tends to destroy the cohesion of the metal across the surface of slip, and in certain cases this develops into a crack. Once a crack is formed it quickly grows in a well-known manner, by tearing at the edges, in consequence of the concentration of stress which results from lack of continuity."

" The experiments throw light on the known fact, that fracture by repeated reversals or alternations of stress resembles fracture resulting from a 'creeping' flaw in its abruptness, and in the absence of local drawing out or other deformation of shape."

The formation of cracks does not appear to be so serious in the case of unidirectional stress, as in the case of alternating stress, there being some especial weakness induced by reason of the alternations. It is a curious fact, that a piece of material after being fatigued by varying stress actions and which has nearly arrived at its limit of endurance, does not appear to show any marked deterioration either in tensile strength or in its plastic properties when tested by the static method. Prof. Ewing says this is not to be expected, so long as the reversals have not been sufficient to produce a crack, but it will have a *yield-point* corresponding to the amount of permanent extension it has received.

7. Fatigue of Metals (*Continued*).—**Prof. Osborne Reynolds' and Dr. J. H. Smith's Method of Testing Metals by Tensile and Compressive Reversals of the Mean Stress.***—In the year 1899, Prof. Reynolds suggested a novel method of testing the "Fatigue of Metals," and gave a general idea of the design of the apparatus and means by which the experiments should be carried out to Mr. J. H. Smith, the 1851 Exhibition Scholar, at his Whitworth Laboratory, Owens College, Manchester. The credit for the details of the design, calculations, and execution of the work has been honourably given by the former to the latter engineer in their joint paper on a "Throw Testing Machine for

---

* See Vol. 199, *Philosophical Transactions of the Royal Society of London*, Series A, November, 1902; also, T. E. Stanton and Leonard Bairstow's paper on "The Resistance of Iron and Steel to Reversals of Direct Stress," in the *Proc. Inst. C.E.* (Paper 3,630), Session 1905-1906.

Reversals of Mean Stress," as read before the Royal Society on March 20, 1902. This arduous research occupied about two years, and the author is indebted to Dr. Smith for his kind permission to reproduce in this book a concise outline of the objects, methods, and results whereby the undertaking was achieved.

Objects of the Research.—"(1) The stress should be direct tension and compression, and (2) of approximately equal amounts, such tension and compression being obtained by means of the inertia force of an oscillatory weight; (3) the rapidity of repetitions should be much higher than in the experiments of Wöhler, Spangenberg, Bauschinger, and Baker; in fact, ranging as high as 2,000 reversals per minute."

Hitherto most of the experiments on "repeated stress" had been carried out on bars subjected to bending, where the ordinary formula for stress in a bent bar was used to calculate the stress at breaking. In such experiments, it had been assumed, that the distribution of the stress at the breaking-down point was the same as for an elastic bar. But, no such assumptions are necessary with this method. Further, the tensile and compressive stresses being nearly of the same value, the elastic limits would soon be changed to their *natural* positions, and the "range of stress" for unlimited reversals would be this *natural elastic range.* If the *limiting range* coincided with the *natural range*, it would be constant whatever the rate of the reversals. Most of the experiments were conducted to find out the *variation of the limiting range of stress* as the rapidity of the reversals increased; but the importance of extending the experiments to very high speeds was fully recognised, in view of the tendency of engineers to adopt fast-speed reciprocating engines, tools, and machinery.

FIG. 13.—THROW TESTING MACHINE FOR REVERSALS OF MEAN STRESS.

The Apparatus and the Method of Applying the Stresses.—The line diagram (Fig. 13) merely indicates the more important parts as far as this concise description is concerned, but it will be understood from it, that the test specimens consisted of round short rods of small diameter. These rods were about $3\frac{1}{2}''$ over all, with screwed ends of $\frac{3}{4}''$ diameter by $1\frac{1}{8}''$ long, turned down in their middle part to $\frac{1}{4}''$ diameter by $\frac{1}{2}''$ long, and then filleted out to short collars the size of the bottom of the threads of their $\frac{3}{4}''$ screwed ends, in a similar way to the test-pieces shown by Fig. 9, but much shorter in the body. The ends of each specimen were screwed home into the top and bottom chucks, or upper and lower spindles, and made
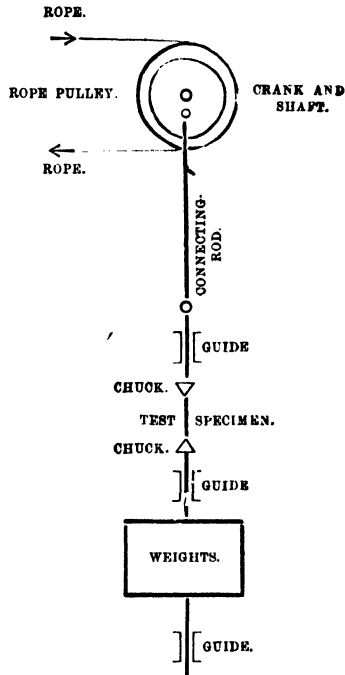
fast by lock-nuts. Upon the lower spindle there were suspended any desired number of flat round disc weights; whilst the top end of the upper spindle was attached to a crank of ¼″ radius or "throw" by a connecting-rod 12″ in length, and the whole truly arranged for vertical reciprocation by the three guide bushes, as shown by Fig. 13. The overhanging crank-pin was turned out of the solid from one end of its crank-shaft. This shaft was very strong and rigid, being supported by two bearings. It was uniformly rotated by a rope and coned pulley drive, the pulley being fixed to the back end of the crank-shaft, which extended beyond the after bearing.

The periodic, vertical, up-and-down motions of the two spindles (between which the test-piece is fixed) are approximately simple harmonic motions, due to the length of the connecting-rod being that of 24 cranks $\left(\dfrac{12''}{.5''}\right)$, and with good lubrication of the guide bearings, the friction was reduced to a minimum. The weight attached to the lower spindle *plus* the weight of the latter produced, by their inertia, *a tension stress at the bottom end* and *a compression stress at the top end of the stroke.* For a fixed load and speed, the stress per square inch of cross-section on test-pieces was varied by inserting specimens of different diameters ; or, the load or the revolutions per minute of the crank could be varied at pleasure to produce different stresses

Uniform Torque, Total Energy, and Balancing of the Moving Parts.—In order to enable the calculations of the stresses in the specimens to be correctly determined, the axis of the crank-shaft must be at rest, whilst the crank revolves with uniform angular velocity. This was effected by driving the crank-shaft with a constant turning effort or "torque," making the total kinetic energy of the moving parts constant for each experiment, and by good all-round balancing of these parts.

The total energy of the moving parts was rendered invariable by *first* attaching a second horizontal connecting-rod to a lug-pin on the large crank-head of the vertical connecting-rod, at the level line of the crank-shaft centre. This second connecting-rod was equal in length to 18 cranks, and gave a reciprocating horizontal motion to adjustable sliding weights (not shown in Fig. 13). Since the vertical and the horizontal weights each received their motions from the same crank-pin, the *velocity* of the one varied as the *sine* and the other as the *cosine* of the angular displacement of the crank.

Hence, the sum of the squares of their velocities were constants, since $\sin^2 a + \cos^2 a = 1$. Consequently, the *total kinetic energy* of the parts were rendered constant when the total weight moving in a horizontal direction was made equal to that moving in a vertical direction. *Second*, by a rigid system of balancing the rotating parts, the whole of the moving parts of the machine were dynamically adjusted.

Determination of the Tensile, Compressive, and Range of the Reversal Stresses, with Formulæ and Results on Mild Steels.—Referring back to Lecture XII., Vol. I., we see, on pp. 314 to 316, the formulæ and figure connecting *centrifugal force* with the *weight* and *acceleration* of a particle moving uniformly in a circle. In the figure (p. 315) alluded to, P A represents the connecting-rod at the top of the stroke. Here r/l vanishes, and we are left solely with the upward momentum of the vertical weight *minus* the dead weight, thus causing *compression*.

Now, when the crank moves round clockwise to its lowest position, r/l also vanishes, and we are again left with the downward momentum of the weight *plus* the dead weight, causing a *tension* stress.

Again, during the time that the crank arm moves clockwise at a uniform angular velocity from its highest to a horizontal position, the vertical

weight moves downwards, from zero velocity to a maximum velocity, with a certain acceleration. Hence, the *rate of change of its momentum* is equal to *the product of its mass* (W ÷ g) *and its acceleration* during that time; or, the force of the moving weight on the specimen is equal to its mass × its acceleration ± its dead weight.

Let r = Radius of crank in feet.
  ,, l = Length of connecting-rod in feet.
  ,, W = Weight of the moving mass in lbs.
  ,, v = Velocity of mass in feet per second.
  ,, ω = Angular velocity in radians per second.
  ,, N = Revolutions per minute.
  ,, n = Revolutions per second = N ÷ 60.
  ,, g = Acceleration due to gravity.
  ,, A = Area of the cross-section of specimen in square inches.
  ,, F_c = Force or stress due to compression in lbs.
  ,, F_T = Force due to tension in lbs.

In the "Throw Testing Machine," let the ratio of the crank radius r to the length of the connecting-rod l be taken into account, and let us note that the acceleration of the moving weight = $\dfrac{v^2}{r}$, whilst its mass is = $\dfrac{W}{g}$.

Then, the *Force of Compression* per sq. in. on the specimen $\Big\}$ = $F_c = \left(\dfrac{W}{g} \times \dfrac{v^2}{r} - \dfrac{W}{g} \times \dfrac{v^2}{r} \times \dfrac{r}{l}\right)\dfrac{1}{A} - \dfrac{W}{A}$ lbs.

Or,   ,,   ,, $F_c = \left(\dfrac{W\,\omega^2 r^2}{g\,r} - \dfrac{W\,\omega^2 r^3}{g\,r\,l}\right)\dfrac{1}{A} - \dfrac{W}{A}$ lbs.,

i.e.,   ,,   ,, $F_c = \dfrac{W\,\omega^2 r}{g\,A}\left(1 - \dfrac{r}{l}\right) - \dfrac{W}{A}$ lbs.

In the same way and from the previous reasoning, we obtain the formula for the tensile stress per square inch on the test-piece when the weight arrives at its lowest position.

Thus, the *Force of Tension* per sq. in. on the specimen $\Big\}$ = $F_T = \dfrac{W\,\omega^2 r}{g\,A}\left(1 + \dfrac{r}{l}\right) + \dfrac{W}{A}$ lbs.

And the *Range of Stress* is equal to the sum of these two stresses.

Or,   The *Range of Stress* = $F_c + F_T = \dfrac{2\,W\,\omega^2 r}{g\,A}$ lbs. per square inch.

Results on Mild Steel.—In order to apply this formula to one or more of Dr. Smith's results, we here quote two of them :—

Oscillatory weight, 12·42 lbs.   Diameter of specimen, ·2469 inch.

SET A. MILD STEEL.	Revolutions per Minute.	Maximum or Tensile Stress per Sq. Inch.	Minimum or Compression Stress per Sq. Inch.	Range of Stress per Sq. Inch.	Number of Reversals before Rupture.
Annealed, . .	2,126	7·99 tons	7·11 tons	15·1 tons	248,700
Unannealed, .	2,122	8·03 tons	7·17 tons	15·2 tons	226,500

Now, applying the previous formula for *range of stress* to the first specimen, we get—

$$F_C + F_T = \frac{2\,W\,\omega^2\,r}{g\,A \times 2,240}\ \text{tons per square inch.}$$

Here, W = 12·42 lbs.; $\omega^2 = (2\pi n)^2 = \left(\frac{2\pi N}{60}\right)^2$; N = 2,126 ;

$r = \left(\frac{·5067}{12}\right)$ feet; $g = 32·18$;

$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(·2469)^2 = ·04788$ square inch ;

2,240 = lbs. per ton.

Hence, $F_C + F_T = \frac{2 \times 12·42}{32·18 \times 2,240}\left(\frac{2\pi}{60}\right)^2 \times \frac{·5067}{12} \times \frac{N^2}{A}\left\{\begin{array}{l}\text{tons per}\\\text{sq. inch.}\end{array}\right.$

Or,   Range of stress = $1·595 \times 10^{-7} \times \frac{N^2}{A}$ tons per square inch.

That is, Range of stress = $(1·595 \times 10^{-7})\dfrac{(2,126)^2}{·04788}$

,,   ,,   ,,   = 15·08 tons per square inch.

This answer corresponds with the value given in the previous table.

Prof. Reynolds, in his preface to the previously-mentioned joint Royal Society paper, says, that Dr. Smith's experiments had brought out evidence of two general laws which had hitherto been unsuspected, viz. :—
(1) That under a given range of stress the number of reversals before rupture *diminishes* as the frequency *increases*.
(2) That the *hard steels* will not sustain more reversals with the *same range* of stress than the *mild steels* when frequency of the reversals is great.
This later statement is borne out by the results given in the previous table, and some of Dr. Smith's other tables show signs of the first law, and he suggests that the explanation of the results will probably be traced to the *hysteresis effect—i.e.*, to an accumulation of the lag which occurs in the elastic range. *

8. Impact Tests.—In addition to what was said in the previous lecture under the heading " Different Mechanical Tests," and to what has just been written under tests 6 and 7 on the "Fatigue of Metals," it may be remarked here that impact tests *are necessary* for steels as well as for other metals which are to be afterwards subjected to *shocking treatment—i.e.*, to rapidly-repeated loads of one kind, or to alternating loads.   Messrs. Seaton and Jude found, that commercial brands of steel, which contained only a small percentage of carbon, gave the best shock-resistance results, and that brittleness was increased very rapidly by an increase in the percentage of

---

* *Mechanical Hysteresis.*—This term has been applied to the area enclosed between the *ascending* and *descending* stress-strain curves.   These curves are obtained by gradually added loads from 0 to a maximum, and decreasing again to zero, in a similar way to the means taken to obtain *magnetic hysteresis* of iron or steel.

carbon. Further, that the line of fracture of the metal followed the direction of the ferrite by avoiding the pearlite. They also observed (as pointed out in the previous lecture under Kirkaldy's inventions), that oil out in the previous lecture under Kirkaldy's inventions), that oil quenching had the extraordinary effect of increasing the shock strength of certain commercial steels some 500 to 600 per cent.

Seaton and Jude's Impact Testing Machine. — This testing machine consists of a weight (of, say, 6 lbs.) being allowed to drop down freely through a distance (of, say, 24 inches) upon the test-bar specimen of the metal in question, as clearly indicated by the accompanying line diagram (Fig. 14). The small specimen of a ∧ notched bar 4″ × ½″ × ½″, where the notch is only ⅛ inch deep, is given by Fig. 15, and its position in the previous figure shows how it may be fixed upon supports having a horizontal distance between them of, say, 3 inches. After each blow the test-piece is reversed, and the apparatus is designed more for the purpose of testing the endurance of a material under a few repeated shocks rather than for testing under what shock it can be fractured by one or two blows. It is preferred to have the test specimens slightly notched, as shown by Fig. 15, because
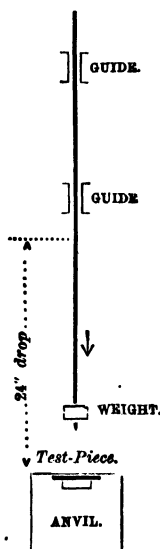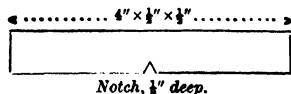
FIG. 14.—LINE DIAGRAM OF "IMPACT"   FIG. 15.—"IMPACT" TEST-PIECE.
TESTING MACHINE.

As used by A. E. Seaton and A. Jude.

most members in engineering structures have scratches, nicks, indentations, or even notches in some cases.

Modern commercial forms of impact testing machines are of two principal types. In the first type, introduced by Mr. Izod, a pendulum breaks off a small notched specimen, and the energy taken out of the pendulum swing is measured. In the second type —such as designed by Seaton and Jude—a hammer gives a number of blows on a notched specimen, and the number of blows before fracture are measured.

Advantages of the Impact Test.—It is affirmed by the above-mentioned advocates of this test :—

(1) That it is a gauge of the preponderant stress or "dislocating agent."

(2) That materials which are sufficiently ductile will resist this impact or shocking test ; but, that certain materials which give excellent tensile and elongation results will not stand this impact test.

(3) That 99 per cent. of all forgings are notched in some way or other, and that internal or external sharp edges cannot always be avoided in the design of machines, or even deep scratches in their construction.

Cumulative Effects of Small and Medium Shocks.—One effect which is sometimes confused with the phenomena investigated by Wöhler, but which should be treated as distinct from them, is the failure which is sometimes brought about through the cumulative effect of shocks. When a blow or shock expends kinetic energy in stressing a material or structure, the strain (which may be more or less general or local according to circumstances) is such, that the work done in producing it is equal to the energy of the blow. If the shock resulting from the kinetic energy of the blow be more than can be absorbed by the elastic property of that portion of the material or structure where it occurs, the elastic limit will be exceeded and local hardening will occur. If such shocks be oft repeated they may finally cause the collapse of the material or structure. This may be shown graphically by means of a stress-strain diagram.

Effect of allowing a Factor of Safety for both Fatigue and for Impact.—It is a matter of controversy among certain civil engineers whether it is necessary to allow for repeated impact shock effects in addition to those of mere fatigue stresses. In bridge work, for example, it is found, that the members become exceedingly large and heavy when both are allowed for, and therefore it is often assumed that, as the stresses are more or less intermittent, the allowance for fatigue alone, sufficiently covers those due to impact shocks. The stresses allowed for in the case of, say, wrought-iron bridge-work are only about 5 tons per square inch, and for steel about 6½ tons per square inch. A period of rest for a fatigue structure most certainly tends to restore its elasticity, but whether it also actually restores the metal to its original strength (without any annealing) is not known. It is certain that even annealing will not bring a material back to its best original or best normal condition when cracks or cleavage have once been started in its molecular structure.

9. Torsional Tests.—The usual arrangements for torsional tests in large tensile testing machines are found to be not only inconvenient, but are sometimes essentially incorrect in principle. Moreover, such machines admit but short specimens. Here, even more than for transverse tests, the use of a separate machine is very advisable. The machine shown by a side elevation and plan in Fig. 16, was used by Prof. Kennedy at University College, London. The twist is applied through a worm wheel, and works with a maximum torque of 4,000 lb.-inches. The test-piece *a* is turned at its ends. centred at one end of the mandril of the worm wheel and at the other end in the boss of the steelyard. Before the stress is applied, the test-piece lies quite freely in the bored holes which centre it. Also, the test-piece is at all times free to move endwise, and is driven from an arm on the mandril through a friction-clutch. A similar friction-clutch at the other end of the test-piece transmits the pressure to the steelyard, and causes it to rise from its fulcrum. An arrangement is made

by which the actual lift of the steelyard end of the test-piece is kept so small as to be negligible. During a test, the steelyard is kept floating by moving the poise-weight outwards along the beam as the twisting moment is increased. A machine capable of twisting asunder an inch bar of steel with a twisting moment of 14,000 to 16,000 lb.-inches is found to be sufficient for most laboratory purposes.
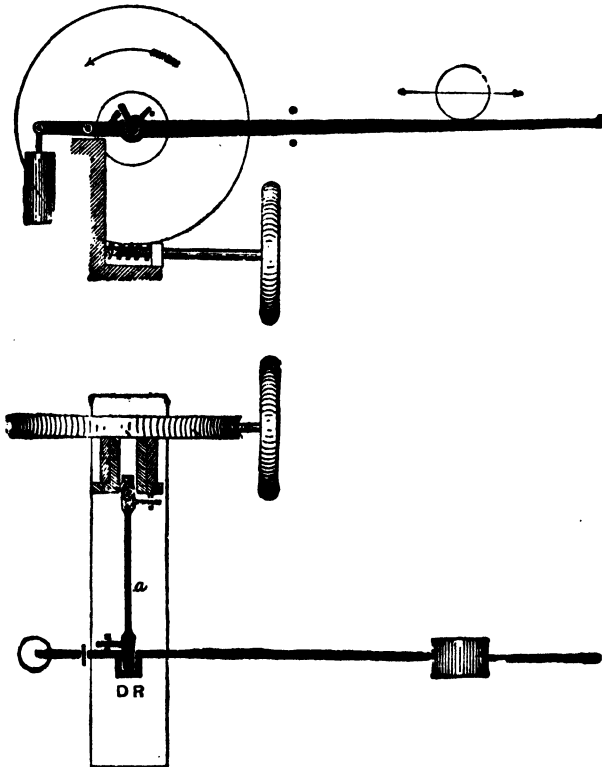
FIG. 16.—MACHINE FOR TORSIONAL STRESSES ON TEST-PIECES. *

Should it be found necessary to record the torsions throughout a test, then an automatic disc recorder, faced with paper and supplied with a spring-pressed pencil, may be fixed at D R, or a plain drum recorder may be used. Specimen torsion diagrams, taken in both these ways, with an example, will be found in the next lecture.

* See *Proc. Inst. C.E.* (Ireland), Nov., 1910, for Prof. Lilly's paper on "A New Torsion Testing Machine."

Torsion of Rods and Wires.—The following Fig. 17 illustrates an arrangement of apparatus for attaching to a wall. The modulus of torsion in rods up to 1 inch in diameter and 2 feet long can be obtained by the aid of the two reading telescopes and mirrors as indicated in the next lecture.
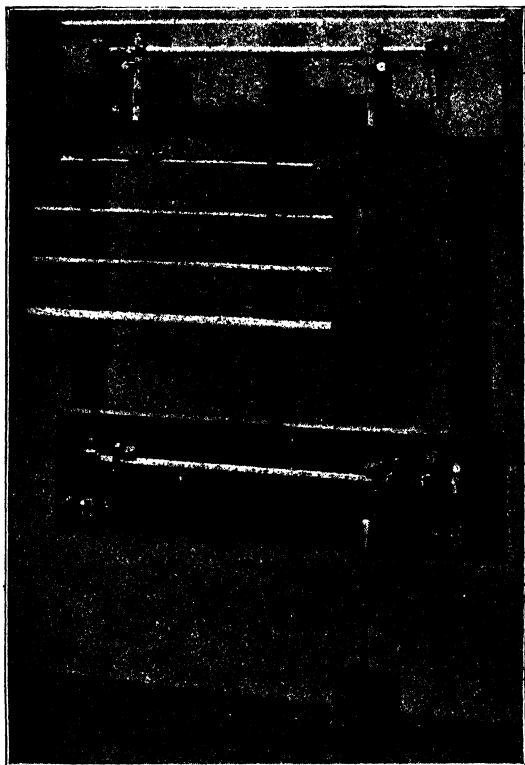
FIG. 17.—APPARATUS FOR MEASURING THE MODULUS OF TORSION IN RODS.

As made by The Cambridge Scientific Instrument Co., Ltd.

Wire Twisting and Bending.—Wires for sheathing submarine cables or ropes are tested, for *twistability*, by taking a specimen from each end of each bundle, and fixing one end of the specimen in the rigid clamp of a hand twisting machine, whilst the other end is gripped by the chuck of the turning handle. The distance between the fixed clamp and the inner end of the chuck may be about 10 inches. A straight black ink line is drawn along the upper surface of the test-piece, and the handle is turned round

slowly until the specimen breaks. The broken parts are brought together and the number of complete spirals or pitch-screw ink thread lines are counted along a length of, say, 6 inches, in order to see whether the specimen has complied with the *torsion test*, as specified in the agreement. The following table shows two of the mechanical tests applied to what is known as " Best Best Annealed Galvanised Iron Wire " for the sheathing of certain submarine cables :—

TABLE OF SIZES, TWISTING AND BENDING TESTS, FOR THE IRON SHEATHING WIRES OF SUBMARINE CABLES.

Weight of Each Wire Bundle.	Diameter of Wire.	No. of Twist in 6 Inches.	Bending and Unbending round a Spindle.
162 lbs.	0·143 inch.	13	0·572 inch.
200 lbs.	0·180 inch.	10	0·720 inch.
280 lbs.	0·230 inch.	6	0·920 inch.
290 lbs.	0·238 inch.	5	0·952 inch.

10. Hardness, Temper and Brittleness.[*]—It is well known, that the physical properties of ductile metals are considerably altered by different kinds of treatment. For example, cold rolling, cold hammering, wire drawing, or stretching will convert a comparatively soft and tough material into a harder and more brittle one. In the case of steel, the physical properties may be altered to a still greater extent by differences in thermal and chemical treatment. The differences in the mechanical properties of steel and other metals are of the greatest practical importance to engineers and metal workers. Under certain circumstances the hardening of a material by special treatment is very useful, such as in the tempering of tool steel. But, in other cases, a material may be quite trustworthy whilst in its normal state and be rendered useless by hardening. Consequently, to get rid of what may be termed "artificially or accidentally induced hardness," processes of annealing are applied after certain mechanical operations.

The ordinary processes of testing by which the tenacity and elongation of a material are measured, indicate indirectly the condition of the specimen, but are open to misinterpretation. For example, a simple tension test of a rail or axle does not clearly discriminate whether the material is properly annealed or not, and, in such cases, a more direct and sensitive test for hardness is required.

· * Students are referred to the "Treatise on Hardening. Tempering, Annealing, and Forging of Steel," by Joseph V. Woodworth, and published by Arch. Constable & Co., Limited, 1903 ; and to the Final Report by Sir F. A. Abel, *Proc. Inst. M.E.*, 1885, on "Steel Hardening," &c.

Definition of Hardness.—Hardness is a property of materials about the definition of which physicists are not agreed. Moreover, they have not specially approved of any one method for accurately measuring or of comparing hardness. Excluding very brittle materials, which crush to powder under pressure, and confining ourselves specially to the metals which are used in engineering constructions, *hardness may be defined as resistance to permanent or plastic deformation.* Consequently, the resistance to indentation by a very hard tool has been taken as a measure of hardness by some experimenters.

Steel containing more than the semi-saturation proportion of carbon ('45 per cent.), when heated to a bright redness and then suddenly cooled, is found to acquire a different kind of hardness from that which is produced by overstraining; for it becomes more brittle and more elastic than before. For instance, if a piece of high-carbon tool steel be heated to a white heat and then plunged into a bath of ice-cold water or mercury, it becomes hard enough to scratch glass like a diamond, and will be very elastic. It is, however, so brittle, that the steel can only be used for such a special purpose as, say, the drilling of tempered steel or chilled iron. Steel treated in this way is called "glass-hard," and entirely loses its plastic character. When this kind of steel is tested for tension it snaps suddenly under mere elastic extension, without any appreciable diminution of its cross-section, and it has but a comparatively small tensile resistance. However, glass-hard steel may be deprived of its brittleness, its strength increased, and the range of its elastic strain greatly increased by reheating it up to a lower temperature and then allowing it to cool slowly. This process is called "withdrawing the temper," and its success or failure depends chiefly on the degree to which the temperature of the steel is thereby raised before slow cooling begins. The different grades of temper which are produced in this way are usually estimated by the colour which appears on the sandstone polished surface of the steel during its reheating. The changes of colour are due to the formation of a varying film of oxide. For example, if the glass-hard metal be reheated to, say, a temperature between 400° F. and 420° F. a light straw-colour appears on its surface. This colour shows that the steel has reached a temper suitable for the points of keen edges in hard cutting tools. Again, raise the heat and hue to, say, a violet-yellow at 500° F., when a temper fit for table knives, where flexibility as well as incisive cutting is required rather than a hard, stiff edge. A temperature of about 550° F. will produce a purple-blue surface and a temper suitable for springs and all articles where the elasticity should be very perfect throughout a wide range of loads. At a temperature of 725° F. the steel passes to a red heat, at which, some kinds of new fast-speed tool steels do the most extraordinary amount of work at a hitherto unsurpassed speed.*

The term *under-tempering* is applied to any process which tends to reduce the hardness of steel to a degree recognisable by the "colour test," and also to all processes by which the degree of hardness is lowered, modified, distempered, or lessened in brittleness. Any process which does not reduce the hardness of steel to a degree denoted by some colour within the range of the recognised "colour test" may be termed *hardening.* The general adoption of a thermometer test as a guide for tempering steels would remove the valid objection to the very old colour test—viz., that

---

* See 6th and later editions of the author's *Manual of Applied Mechanics* for the treatment and capabilities of this new tool steel.

the colour obtained on the piece of steel through heating it is no true indication that it possesses a temper above its natural degree of hardness; because steel, wrought iron, and cast iron will assume all the colours of the "colour test" when polished and heated to the necessary temperature. Thus, the colour which appears on the surface of a piece of steel is simply an indication that it has been heated to a certain temperature and not that it has been actually tempered, or, in fact, that the heating has in any way changed its hardness or its softness!

Workshop Test.—A mechanic tests hardness by using a file, and the earliest scientific scale of hardness was that proposed by Moh. He selected ten substances and arranged them in order, so that each would be scratched by the substance next above it, and would scratch that substance next below it. This simple "scratch test" has been found very useful in the case of brittle bodies; but it is less useful for ductile bodies.

Turner's Scratch Test.[*]—Mr. Thomas Turner has introduced a more definite and exact scratch test of hardness. A diamond point is balanced at the end of a lever and loaded till it just definitely scratches the surface. The load necessary to produce a scratch is taken as the measure of hardness or hardness number. This method requires considerable skill, and it does not seem sensitive enough to discriminate the quality of ductile metals, such as iron and mild steel.

Calvert and Johnson's Test.—Some experimenters have used an abrasion test in determining the hardness of a metal, but this method is again difficult and laborious, and appears to be best adapted for brittle substances, such as stones. Consequently, the only method which has been much used in practice is an indentation test. Messrs. Calvert and Johnson took, as their measure of the hardness of alloys, the weight which caused a small truncated cone to indent the alloy to a depth of 3½ millimetres in half an hour. Whilst, in some tests performed in the United States, the volume of indentation produced by a pyramidal point loaded with a weight of 10,000 lbs. was taken as a measure of hardness. An indentation of ¼ cubic inch was taken as unit hardness, and half that indentation was called hardness number 2·0.[†] This is a much more definite and scientific method of determining hardness in materials which are ductile or plastic.

Unwin's Test.[‡]—The apparatus devised and used by Prof. Unwin, F.R.S., is indicated by the line diagram (Fig. 18). It consists of a loosely-fitted plunger sliding down vertically in a cast-iron guide-block, and pressing an excessively hard ◇ or "indenting tool" placed upon the test-
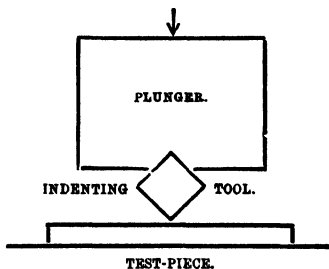
FIG. 18.—PROF. UNWIN'S TEST FOR HARDNESS.

---

* See *Philosophical Magazine*, 4th Series, vol. xvii., p. 114.
† See *Reports on Metals for Cannon*, 1856, Ordnance Department, U.S.A.
‡ See *Proc. Inst. C.E.*, vol. cxxix., p. 336, for paper on "A New Indentation Test for Determining the Hardness of Metals," by Prof. Unwin, F.R.S., &c.

piece.* The whole apparatus is placed under an ordinary testing machine, whereby loads are applied, and the indentation corresponding to each increase of load is noted. Each load is allowed to rest on the specimen for a few minutes, until the indentation ceases to increase. In no case must the specimen be stretched during the experiment.

Comparison between Prof. Unwin's Indentation Method and Turner's Scratch Method of Determining Hardness.—The following table contains the hardness numbers for the same specimens, as determined by the *indentation method* and by the *scratch method*. In order that they may be easily compared, the relative hardnesses have been calculated by taking copper as unity. It will be seen, that the relative hardness follows the same order by whichever method it is determined. But, the scale of hardness with the indentation method is an opener scale than that with the scratch method. Further, the indentation method is found to be easier, more definite, and requires less skill. Of course, for brittle substances the scratch method is more suitable.

HARDNESS OF METALS BY INDENTATION AND SCRATCH METHODS.

Materials.	Hardness Number.		Relative Hardness. Copper = 1.	
	Indentation Method.	Scratch Method.	Indentation Method.	Scratch Method.
Cast steel (normal), .	554·0	25·0	8·94	4·17
Brass (No. 1), .	221·0	12·0	3·57	2·00
Mild steel (normal), .	143·5	9·0	2·32	1·50
Copper (annealed), .	62·0	6·0	1·00	1·00
Aluminium (squirted),	41·8	4·0	·67	·66

Brinell Hardness Test.—In this machine a hardened steel ball is pressed with a pre-determined force against the plate of which hardness is required. The diameter of the resulting curved depression is then found, and from this the "hardness number" is obtained by the formula :—

$$H = \frac{P}{\frac{\pi D}{2}(D - \sqrt{D^2 - d^2})}.$$

P = load.
D = diameter of ball.
d = diameter of depression.

---

* "The indenting tool is simply a short bar of square tool steel, ground accurately, so that the angles are right angles and as hard as possible. Any one of the edges may be used as an indenting edge, and the tool is easily re-ground. The test-piece is a square bar of the metal to be tested usually ⅜ inch square and about 2½ inches long. A scale and vernier are attached to the guide-block for measuring the indentations."

The following table gives values :—

BRINELL'S HARDNESS NUMBERS FOR LOAD 3,000 KG.).

Diameter of Steel Ball = 10 mm.

Diameter of Ball Impression	Hardness Number	Diameter of Ball Impression	Hardness Number	Diameter of Ball Impression	Hardness Number	Diameter of Ball Impression	Hardness Number
mm.		mm.		mm.		mm.	
2·0	946	3·25	351	4·50	179	5·75	105
2·05	898	3·30	340	4·55	174	5·80	103
2·10	857	3·35	332	4·60	170	5·85	101
2·15	817	3·40	321	4·65	166	5·90	99
2·20	782	3·45	311	4·70	163	5·95	97
2·25	744	3·50	302	4·75	159	6·0	96
2·30	713	3·55	293	4·80	156	6·05	94
2·35	683	3·60	286	4·85	153	6·10	92
2·40	652	3·65	277	4·90	149	6·15	90
2·45	627	3·70	269	4·95	146	6·20	89
2·50	600	3·75	262	5·0	143	6·25	87
2·55	578	3·80	255	5·05	140	6·30	86
2·60	555	3·85	248	5·10	137	6·35	84
2·65	532	3·90	241	5·15	134	6·40	82
2·70	512	3·95	235	5·20	131	6·45	81
2·75	495	4·0	228	5·25	128	6·50	80
2·80	477	4·05	223	5·30	126	6·55	79
2·85	460	4·10	217	5·35	124	6·60	77
2·90	444	4·15	212	5·40	121	6·65	76
2·95	430	4·20	207	5·45	118	6·70	74
3·0	418	4·25	202	5·50	116	6·75	73
3·05	402	4·30	196	5·55	114	6·80	71·5
3·10	387	4·35	192	5·60	112	6·85	70
3·15	375	4·40	187	5·65	109	6·90	69
3·20	364	4·45	183	5·70	107	6·95	68

For other test loads, the hardness numbers are proportional
to those in the table.

Within certain limits the Brinell Hardness Number of a steel is roughly proportional to its tensile strength.

For the most recent information upon hardness of metals, see the report of the Special Committee of the Institution of Mechanical Engineers (*Proc. Inst. M.E.*, 1917).

Caledonian Railway Company's (1908) Specification for Steel Rails, 90 lbs. per yard Section.*

1. *Weight of Rail.*—The weight of the rail shall be 90 lbs. per lineal yard.

2. *Chemical Composition.*—The steel for the rails shall be of the best quality made by the Bessemer, Siemens-Martin, or other process, to be approved by the company's engineer.

The rails shall show on analysis that in chemical composition they conform to the following limits:—

Carbon, .	.	.	.	.	From 0·35 to 0·50 per cent.
Manganese,	.	.	.	.	,, 0·70 ,, 1·0 ,,
Silicon, .	.	.	.	.	Not to exceed 0·1 ,,
Phosphorus, .	.	.	.		,, ,, 0·075 ,,
Sulphur,	.	.	.	.	,, ,, 0·08 ,,

3. *Chemical Analysis.*—The manufacturer shall make and furnish carbon determinations of each cast to the representative of the company.

A complete chemical analysis, representing the average of the other elements contained in the steel, shall be similarly given for each rolling. Such complete analysis shall be made from drillings taken from the rail, or tensile test piece or pieces. When the rolling exceeds 200 tons, an additional complete analysis shall be made for each 200 tons, or part thereof.

Should the company desire to make independent chemical determinations, the necessary specimens and samples shall be furnished by the manufacturer. For this purpose not more than two rails in every hundred tons manufactured shall be selected by the company's engineer, and drillings taken from the head of the rail, unless otherwise specified by him. If, upon being subjected to the specified tests, either sample fails to comply therewith, then all the rails in the blow, of which the test pieces form a part, may be rejected; and the representative may take similar samples from other two rails out of the same 100 tons, and, should either fail to comply with the specified analysis, the whole 100 tons may be rejected.

In case of difference between the company and the manufacturer as to the accuracy of an analysis, either party shall have the right to have samples of the steel analysed by an independent metallurgist, to be mutually agreed upon. The expenses attendant upon such independent analysis shall be borne by the party adjudged to be in the wrong.

4. *Manufacture.*—Each rail shall be made from an ingot not less than 12 inches square at the smaller and 14 inches square at the larger end. The ingot must be cogged down into blooms, and sufficient crop then sheared from each end to ensure soundness.

5. *Permissible Variation in Weight.*—A rolling margin of ½ per cent. under to ¼ per cent. above the specified weight will be permitted, but the contract weight only will be paid for.

6. *Templates.*—Before the general manufacture of the rails is commenced the company will supply the manufacturer with a set of templates, internal

---

* I am indebted to D. A. Matheson, Esq., M.Inst.C.E., General Manager of the Caledonian Railway Company, for permission to print this Specification. It is drawn out on the same lines as the "British Standard Specification for Bulb and Head Steel Rails." See *Note* on p. 76 of this Vol. II.—A. J.

and external, and the manufacturer shall then make a set of templates which shall be submitted for the approval of the company's engineer. Each template shall have neatly engraved upon it "C. R. 90 lbs. per yard," the manufacturer's name and address, and the date of the contract.

7. *Rails to be free from Defects.*—The whole of the rails shall be of uniform section throughout, with the ends cut true and square, true to template, perfectly sound and straight, and free from splits, cracks, burrs, and defects of every kind. All straightening shall be done by pressure and not by hammering.

8. *Length of Rails for Straight Line.*—The rails shall be 48 feet in length at a temperature of 60° Fahr., but a quantity of short lengths will be taken in such lengths and quantities as may be ordered by the Company, provided that these short lengths are cut down from longer lengths found to be defective at the ends only, and that the total quantity taken does not exceed $7\frac{1}{2}$ per cent. of the contract.

9. *Permissible Variation in Length.*—No rail will be accepted which is more than three-sixteenths of an inch ($\frac{3}{16}''$) above or below the length specified, whether for straight or curved line.

10. *Rails of Special Length for Matching in Curved Line.*—When required by the company, rails are to be supplied $4\frac{1}{2}$ inches shorter than the normal length to match in curves, and these special lengths are to be distinctly painted in white paint for a length of about one foot at each end.

11. *Rails for Switches and Crossings.*—Rails are to be supplied for switches and crossings when so ordered, and such rails are to be in lengths of 12 feet, 15 feet, 18 feet, 21 feet, or 24 feet, cut from sound long rails, and drilled at one end for fish plate bolts.

12. *Branding.*—The company's initials "C. R.," the manufacturer's name, and the month and year of manufacture shall be rolled in letters three-quarters of an inch ($\frac{3}{4}''$) in size on one side of the web of each rail; and the number of the cast or blow from which it has been rolled shall be stamped on the end of each rail in half-inch ($\frac{1}{2}''$) block figures.

13. *Impact Test.*—The company's engineer shall select one rail from each cast from which a length of 5 feet will be cut for testing under the falling weight. Each 5 feet length of rail selected for testing shall be placed horizontally, with the bull head upwards, upon two iron or steel supports having a solid foundation, and placed so that their centres shall be 3 feet 6 inches apart, the rail bearing of such supports being rounded to a radius of 3 inches. It shall then be subjected to two blows from a falling iron weight of 2,240 lbs., the radius of the underside of which is to be 5 inches. These blows must be sustained without fracture, and the deflection should not exceed from $\frac{7}{8}$ inch to $1\frac{1}{4}$ inches after the first blow due to a fall of 7 feet, or from 3 inches to $4\frac{1}{4}$ inches after the second blow due to a fall of 20 feet.

Should the length cut from the selected rail fail to comply with the specified test, two other rails from the same cast will be selected and similar lengths cut and tested. The acceptance or rejection of the cast will be decided by the result of the three tests, so that, if two of the rails selected fail to comply with the test, the entire cast will be rejected.

14. *Tensile Test.*—From each 100 tons of rails the manufacturer shall cut a test piece from any rail selected as a sample rail, such test piece to be stamped to correspond with the sample rail. It shall then be placed in a testing machine of approved pattern, and shall show an ultimate tensile strength equivalent to not less than 40 tons per square inch nor more than 48 tons per square inch, with an elongation of not less than 15 per cent. upon the British Standard tensile test pieces C or D. Should

the test piece fail to fulfil these conditions, the company's engineer may require the manufacturer to test another rail from the same cast in a similar manner, and if the second rail fails to comply with the specified requirements, the whole of the rails rolled from that cast may be rejected ; and the company's engineer may take similar samples from a further two rails out of the same 100 tons, and should either fail to comply with the specified tensile strength and elongation, then the whole 100 tons may be rejected.

Should the company desire to have independent tests made, the manufacturer shall provide the necessary test pieces, viz., two for every 200 tons, properly shaped and prepared as described for the British Standard Tensile Test.

15. *Holes in Rails.*—The holes for fishbolts must be drilled through the web from the solid at each end of the rails, of the sizes and in the position shown on a drawing to be supplied by the company. These holes must be clean and square with the web, without burrs on either side, and will be checked with the gauges to be furnished to the manufacturer by the company. Should any of the holes vary from the correct size or position more than one thirty-second of an inch ($\frac{1}{32}''$) the rails in question will be liable to rejection.

16. *Notice of Rolling to be given.*—The manufacturer shall give to the company's engineer at least seven clear days' written notice before commencing to roll the first lot of rails, and at least three clear days' written notice before commencing to roll any subsequent lot of rails, in order that arrangements may be made for the presence of the engineer or his representative at the rolling.

17. *Inspection and Testing.*—The company's engineer shall be at liberty to examine the rails during any state of their manufacture, and shall have access at all reasonable times to the works where they are to be manufactured. Every facility is to be given, and all labour, materials, and appliances are to be provided free of charge by the manufacturer for the inspection and such testing of the rails as takes place at the works.

Before the rails are submitted to the company's engineer for inspection the manufacturer must have them examined, and all rails which he admits to be defective are to be sorted out and placed in a separate stack ; the company's engineer being empowered to refuse to inspect any lot of rails not put in uniform lengths and sorted.

18. *Marking of Accepted Rails.*—All rails accepted by the representative of the company's engineer shall be stamped in his presence.

LECTURE VI.—QUESTIONS.

1. Explain why designers of torpedo boats aim at giving the hulls of these very fast short vessels as uniform elasticity as possible throughout their length, in addition to a maximum of strength with a minimum of weight. State how they accomplished this object.

2. State your views concisely *re* shearing *versus* planing and punching *versus* boring of low-grade and of high-class plates for engineering structures. Show, by sketches, &c., how the ultimate cost of boring such plates may be minimised. Also, how both shearing and punching affect the condition of mild and the harder kinds of steel plates.

3. What objections can you offer to the punching and subsequent drifting of holes in boiler and other steel plates which have to be riveted together ?

4. Why is it of importance to have reliable results on the "fatigue of metals" which are used in certain machines and structures ?

5. Mention a few of the chief experimenters upon the fatigue of metals with dates, and give, in concise general terms, the results of their work.

6. Explain as concisely as you can, the methods adopted by Wöhler and the deductions derived from his experiments.

7. Give Gerber's formula as based on Wöhler's experiments, and explain the same in detail.

8. Explain in your own words Bauschinger's deductions on the "elastic limits" of materials.

9. How is heating produced in a metal when it is subjected to severe stresses ? How do rest, re-heating, and annealing relieve the strains of a stressed metal bar.

10. Explain concisely Prof. Ewing's conclusions *re* the inter-molecular crystalline action of metals subjected to severe stresses and fatigue.

11. Explain, by aid of a sketch, the action of the machine, as suggested by Prof. Reynolds, which was designed and used by Dr. J. H. Smith in carrying out a series of experiments on the tensile and compressive reciprocating reversal stresses on round, short test-specimens of iron and steel.

12. Explain how the formula is derived and applied to the machine and test-pieces of the former question, by using the data for the second example in the text on unannealed mild steel.

13. Explain, by aid of line diagram figures, how Messrs. Seaton and Jude's impact tests for notched and unnotched bars are carried out. State the necessity for and the advantages of this test for certain materials, as used in certain machines and structures. State why ordinary tensile, compressive, and other tests do not give the same direct useful information which impact tests afford for such cases.

14. Explain, by aid of simple sketches, a form of torsion testing machine for round bars.

15. How are samples of wire for ropes and for the sheathing of submarine cables tested ? Give a sketch of the machine you would propose to carry out such tests, and what would happen if the contractor's man drew a straight ink-line along the upper and the lower surfaces of the wire when it was tested ? If you were not present during the test, how would you detect the mistake ?

16. How would you test plates or bars of steel for hardness, and how would you register your results ?

17. How would you test a metal for brittleness ? What results would you expect in the case of a very brittle specimen ?

LECTURE VII.

THE MEASUREMENT OF STRAIN.

Autographic Apparatus for Recording Stress - Strain Dia-grams. —In laboratory testing, the relation of extension beyond the elastic limit to the load throughout the test, may easily be observed by applying a pair of beam compasses to two punch-marked points on the test-piece from time to time as the test proceeds, and transferring the distance to a scale upon squared paper. The original distance between the two points is usually from 8 to 10 inches. Or, Prof. Kennedy's mechanical extensometer may be used, as seen from Fig. 19. When testing is to be done quickly, and, since a knowledge of this relation is wanted afterwards, some form of autographic recording apparatus is found to be very convenient.

In most of the arrangements which are used in practice, the diagram is drawn by the relative movement of a pencil and a sheet of paper wound on a drum, in the same way that indicator cards are taken from the cylinders of steam engines. The one component of the motion is proportional to the extension, and the other component to the travel of the weight by which the load is measured.
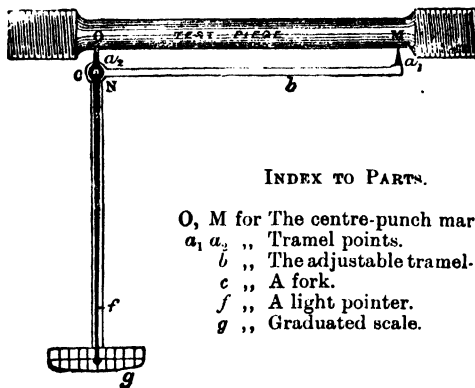
INDEX TO PARTS.

O, M for The centre-punch marks
$a_1 a_2$,, Tramel points.
b ,, The adjustable tramel-rod.
c ,, A fork.
f ,, A light pointer.
g ,, Graduated scale.

FIG. 19.—PROF. KENNEDY'S MECHANICAL EXTENSOMETER. "

In the single-lever vertical testing machine, as shown by Fig. 6, a mechanical autographic recorder is attached.
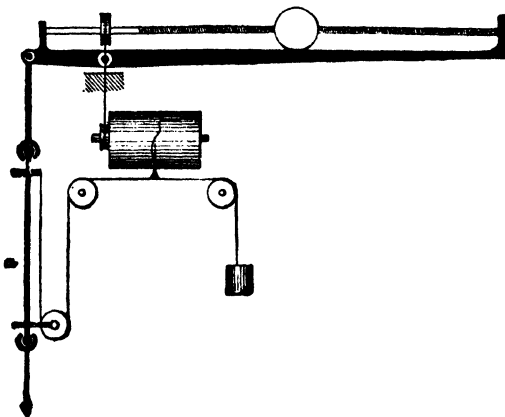
FIG. 20.—PROF. UNWIN'S AUTOMATIC TEST-RECORDING APPARATUS.

* Figs. 19 to 26 and 33 are from Prof. Kennedy's paper on " Engineering Laboratories," by his kind permission and by the favour of The Institution of Civil Engineers. See *Proc. Inst. C.E.*, vol. lxxxviii.

In the above Fig. 20 Prof. Unwin's recorder is shown supporting the paper drum horizontally on the main column. The paper drum is driven round synchronously with the rotation of the screw which moves the travelling poise-weight along the weight-beam. The extension of the test-piece is taken by having two clips firmly secured to the test-piece at points, say, 8 inches apart, by a fine wire or inextensible cord attached to the upper clip and passing under a pulley on the lower clip, then over a pair of pulleys parallel to the paper drum. The pencil is attached to the wire between the last pair of pulleys and traces on the paper of the drum the stress-strain diagram, as clearly indicated by the figure.
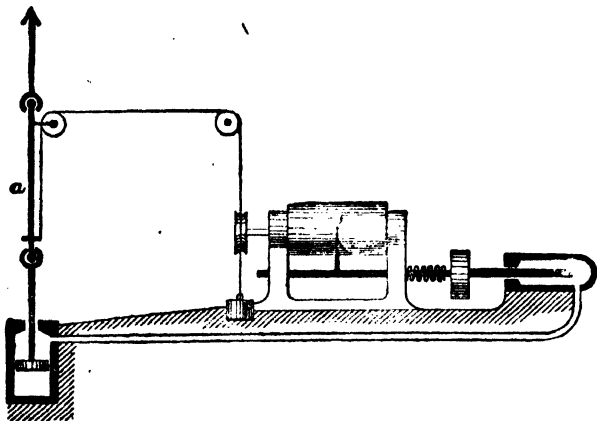
FIG. 21.—WICKSTEED'S AUTOMATIC TEST-RECORDING APPARATUS.

In the hydrographic recorder, as shown by the accompanying Fig. 21, and which can only be used with machines having a hydraulic pressure ram, the drum is pulled round by a wire from the test-piece, through distances proportional to the extension, and the pencil takes its motion, not from the travelling poise-weight, but from the piston of an auxiliary hydraulic cylinder in free communication with the hydraulic cylinder of the machine. This piston compresses a spring in its advance, consequently its displacement measures the force with which it is pressed out. The friction of the piston is eliminated by keeping it in continuous rotation, and this is found to make it indicate correctly, the pressure in the main hydraulic cylinder acting on the specimen. The net load on the test-piece does not bear a definite relation to the pressure in the main cylinder owing to the friction of the ram. A uniform scale can, however, be found, because the friction of the ram is proportional to the pressure, and the values can be interpreted by occasional reference to the weigh-beam.

Hence, we get a record of the point of first permanent set and extension at maximum load, while a stress-strain diagram is automatically drawn, showing the behaviour of the test-piece in yielding to the stresses imposed upon it from the start to the finish of the test.

Prof. Hele Shaw's Apparatus for Automatically Drawing Stress-Strain Diagrams.

—The essential principles of this apparatus are identical with those of Mr. Aspinall and Professor Unwin. A front and a side elevation are given in Fig. 22, whereby its action will be easily understood. The front elevation shows a portion of a Wicksteed single-lever testing machine, to which Prof. Shaw's apparatus is attached.
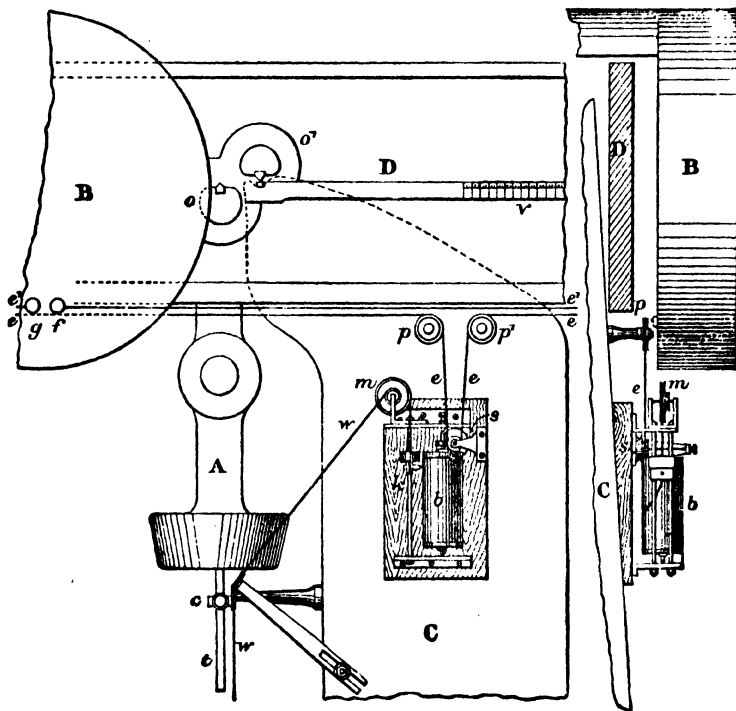
FIG. 22.—PROF. HELE SHAW'S APPARATUS FOR AUTOMATICALLY DRAWING THE STRESS-STRAIN DIAGRAM OF A TEST-PIECE.

It consists of a cord fixed at the two points g and f to the jockey weight B. This otherwise endless cord passes round a pulley at one end of the lever D, then round a pulley p, and finally (by means of another pulley s, on the axis of which is a worm) it turns the barrel b with the paper on which the diagram is to be drawn. The cord then passes away to the other end of the lever and is led back to the other point of attachment. Thus, inasmuch as the position of the weight on the lever gives the exact measure of the force on the specimen by means of a scale v, the distance turned through by the barrel b is directly proportional to the stress on the test-piece t. The strain is recorded in the usual way, for one end of a wire w is attached to one of the points on the test-piece, passes over the

pulley *c*, and finally to a small pulley on the same spindle as the larger pulley *m*. Another wire passes over the larger pulley *m* and carries the recording pencil *k*, whilst the tension on the wire is maintained by a suspended weight shown above *k*. Consequently, we have two motions at right angles to each other :—(1) The motion of the barrel about its axis, which gives a measure of the stress, and (2) the motion of the pencil up and down, which gives the measure of the strain. It might at first be thought that the possible stretching of the cord would lead to an incorrect result, but no one will think so when reminded, that the cord travels 15 feet for a 6-inch travel of the barrel, and therefore any possible error from this cause is reduced 30 times by this means. The side view of Fig. 22 shows the barrel *b* in position, and how the jockey-weight B clears the apparatus.

Prof. Kennedy's and H. G. Ashcroft's Instrument for obtaining Stress-Strain Diagrams.—*Principle of the Instrument.*—It will be seen from Fig. 23, that the test-piece *a* is placed in the machine in *series* with a stronger bar *b*, called a "spring-piece." The two bars are connected directly by a simple coupling, and are pulled simultaneously, the one through the other. The spring-piece is of a material such that its limit of elasticity occurs only at a load greater than that which will break the test-piece. It must also be of a perfectly elastic material, as ascertained by previous experiment, so that its extension is strictly proportional to the pull on it, and therefore to the pull on the test-piece. A very light pointer *c* is made to swing about an axis through an angle proportional to the extension of the spring-piece, and to the pull on the test-piece. The end of this pointer always touches a sheet of smoked glass *d*, to which a travel in its own plane is given proportional to the extension of the test-piece.
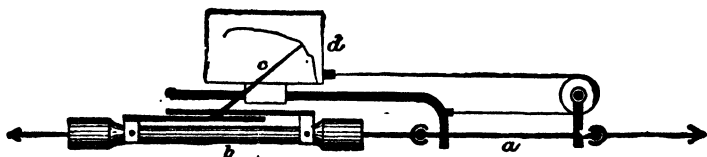
FIG. 23.—PROF. KENNEDY AND ASHCROFT'S INSTRUMENT FOR OBTAINING A STRESS-STRAIN DIAGRAM.

In this way the diagram is drawn. By an arrangement of differential levers it is assured that the motion of the glass depends solely on the extension between the marked points on the test-bar, so that no amount of extension in the coupling, or ends of the test-piece, or in any other part of the apparatus, can move the glass. Also, the instrument is so arranged that the absolute elongation of the spring-piece does not cause any motion of the pointer relatively to the glass.

Advantages and Disadvantages of the Instrument.— It cannot be said to be suitable for general use in works testing; but for laboratory testing, where the instrument is in skilled hands, and not subject to rough usage, it is believed to be, on the whole, more sensitive and to give better diagrams than any of the other forms. It has also the advantage, that it is entirely independent of either the poise-weight or the ram, or even any part of the framing of the testing machine. Its hollow, circular, tapered aluminium pointer may be made so light, that the diagram can be assumed to be free

from any errors due to inertia. The diagrams drawn by this instrument have the drawback, that their load-ordinates are arcs of circles instead of straight lines, and Prof. Kennedy said, in his Inst. C.E. paper on "The Equipment of Engineering Laboratories," that he had not yet succeeded in devising a plan for getting rid of this difficulty without unduly interfering with the sensitiveness of the instrument.
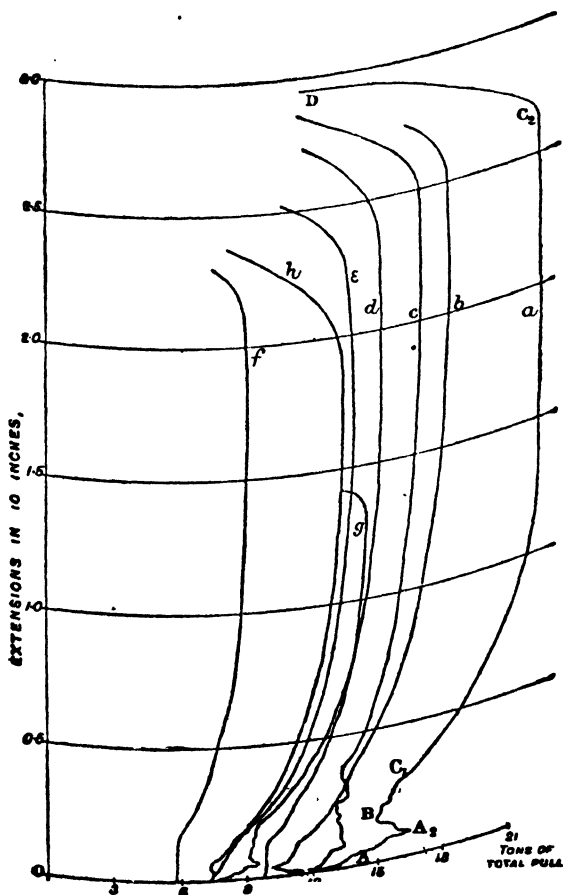
Fig. 24.—Automatically Drawn Stress-Strain Diagrams.

By the Kennedy and Ashcroft Instrument.

Swedish bar iron, *a*; Shelton bar iron, *b*; Swedish iron, *c*; Landore rivet steel, *d*; Landore plate, *e* and *f*; cast steel, *g*; mild steel bar. *h*.

Automatically Drawn Stress-Strain Diagrams, giving the Elastic, Working, and Ultimate Strengths of a Material.—It is well known, that the life of most ductile materials, such as wrought iron and mild steel, when of uniform section, and placed in a testing machine with the load gradually increased, will be represented by diagrams similar to those indicated in Fig. 24.

Here, O A₁ represents the very small elastic extensions which are strictly proportional to the stress, and therefore correspond to some constant value of the modulus of elasticity. Should the stress be removed from the test-piece at any time, during the elastic stage, the test-piece will be found to return to its original length.

The stage A₁ A₂ is not always present in test-pieces. In this stage, the strain increases faster than the stress. If the load be removed during this part of the experiment it will be found, that the test-piece does not return to its original length, but it will have received a small amount of permanent set.

A₂ is the point which is called the "*break-down*" point; or better still, the "*yield-point*," where the character of the material changes.

A₂ B shows a remarkable feature; for the apparent fall or "*release*" of stress of about 2 tons which takes place after the yield-point, is accompanied by a sudden rapid extension, without any increase of the load.

C₁ C₂ represents the main part of the stretch of the bar, while it is in a condition of "*uniform flow*" or of *semi-plasticity*. It is found from observations, that the actual extension of the test-piece in this stage, depends partly on the time during which the load acts. Also, that the extension increases with time even when the load is not increased.

C₂ shows the point of maximum load, usually termed the *breaking load*.

The curve C₂ D shows the decrease of stress as the bar draws down or flows locally. The test-piece is now ruptured at the point D, which is called the *terminal load*.

The stage A₁ A₂ is sometimes absent or minute. A₂ B C₁ takes various forms; and C₂ D also takes very various forms, or may be absent altogether. But a complete test-recorder ought to be able to show all the stages accurately, and where they exist. It has been said, that the stages C₂ D and A₂ B are not of any practical importance. It is quite possible, that for works testing, both these stages may be neglected, without the loss of much valuable information. But, it should not be overlooked, that the former gives very much the same information about any material as is obtained by measuring its reduction of area. It seems inadmissible to neglect both or either of these stages in scientific experiments, not only for the sake of accuracy, but because there is by no means a full knowledge of the practical bearing of either of these stages. In general practice, the *yield-point* is taken as the *elastic limit of the material*, and is the limit of stress in tons per square inch.

Elastic-Strain Diagrams.—The difficulty in the way of obtaining a proper diagram of the elastic part of a stress-strain diagram lies in the fact, that the extensions for ordinary test-pieces of 10 inches in length must be magnified at least 100 times, in order that the curve may be useful. Professor Unwin devised for this purpose a "semi-automatic" apparatus, which, while it did not actually draw a diagram, it determined a number of points for the curve. Professor Kennedy improved and modified this diagramming apparatus, to give the desired curve about 150 times its full size upon smoked glass. The spring-piece shown in conjunction with the test-piece in Fig. 23 is not now used, but the tracing pointer is placed on the test-piece and used to measure its exten-

sions. The frame is also carried by the test-piece, and the smoked glass is moved by the poise-weight. It requires very careful manipulation to ensure, that the glass starts at the instant at which the test-piece begins to stretch, for any inaccuracy in this matter spoils the starting-point in the diagram, although it leaves the rest of the curve unaffected. The ordinates which in this case represent extensions are curved, like those given in the stress-strain diagram (Fig. 24), and require rectification before the real nature of the curve can be seen. Although the apparatus is thus by no means in an ideal form, yet the diagrams (Fig. 25) which it gives are at least accurate and automatically drawn. The diagrams give a record of facts which are of great interest, but which have hitherto been known only as the result of long and laborious detail measurements and plotting on squared paper.
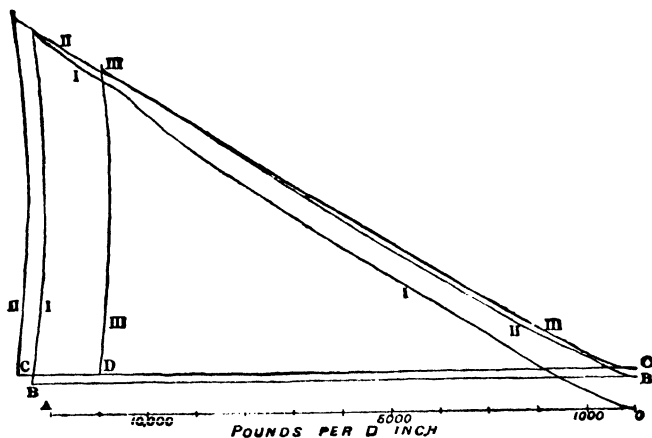
FIG. 25.—AUTOMATICALLY DRAWN ELASTIC-STRAIN DIAGRAMS.

Notes.—These diagrams were taken from a piece of Blaenavon cast iron, ·75 inch in diameter and 10 inches long.
Curve I. was first drawn. It may be observed, that this test-piece shows considerable set after about 10,000 lbs. per square inch had been applied. It comes back finally to a new zero line, on removing the load, and shows a set A B of about 1 + 750 inch.
The load was at once re-applied and gave Curve II., which finally almost runs into Curve I., but shows a small set B C on the removal of the load.
Curve III. was then drawn, and it almost coincides with II., and finally goes back to the same zero line at the point D, thus showing no further set. The diagram exaggeration of the extension is 130 to 1. The small continuations of the extensions at the end of each experiment apparently show a "time effect."

Plasticity.—It is important not only to ascertain the breaking strength of the material to be used in a structure which is to be subjected to live loads and shocks, but also to determine its capabilities to resist deformation without rupture. In materials such as wrought iron and mild steel plasticity is combined with a high tensile strength, but with cast iron and hard steel this property is absent. Consequently, we find it usual in specifying for wrought iron and steel to require a certain percentage of elongation and contraction of area, as well as a certain breaking stress of so many tons per square inch. The plasticity of a material is measured by its final elongation and contraction of area. As previously stated, the elongation is taken on a length of 8 to 10 inches of the test-piece.

To Obtain a Measure of the Work Done per Cubic Inch by aid of the Autographic Diagram, when subjecting a Test-Piece to a Tensile Stress.

—Remembering that work done is the product of the force into the linear distance through which it acts, and that the ordinates on the stress-strain diagram represent to scale the loads in tons while the extensions are measured as abscissæ; then the whole area of the stress-strain diagram represents, to some scale, the work done upon the test-piece in fracturing it. Thus, if x inches on the vertical = 1 ton, and y inches on the horizontal = 1 inch of extension of the test-piece, we get—

$$\text{Work done in inch-tons} = \frac{\text{area of stress-strain diagram in square inches}}{x\,y}.$$

The area of the diagram can be found by means of Simpson's rule or by a planimeter, and if we divide the work done in inch-tons by the volume of the test-piece in cubic inches, we obtain the work done per cubic inch.

$$\text{Work done per cubic inch} = \frac{\text{work done in inch-tons}}{\text{volume of test-piece in cubic inches}}.$$

In order to compare results for the work done per cubic inch in breaking a bar of any material, the test-pieces should be similar, because the ultimate elongation depends on the original length, and to some extent upon the cross-sectional area for the same material.

Prof. Kennedy's Approximate Method.—Prof. Kennedy has pointed out, that the curve during the ductile and plastic stages is a very close approximation to a parabola. Consequently, if we assume it to be so, the work done in inch-tons per cubic inch upon the test-piece can be calculated without the aid of a diagram.

Let f_t = Elastic limit stress in tons per square in .

,, $f_{max.}$ = Maximum stress in tons per square inch.

,, L = Length of the bar between the datum points in inches before extension commences.

,, l = Extension in inches.

Then, the Work done in inch-tons per square inch of cross-section of the test-piece $\Big\} = f_t\, l + \frac{2}{3}\,(f_{max.} - f_t)\, l$

,,   ,,   ,,   ,, $\Big\} = \dfrac{l}{3}\,(f_t + 2 f_{max.}).$

∴ Work done in inch-tons per cubic inch of the test-piece $\Big\} = \dfrac{l}{3\,L}\,(f_t + 2 f_{max.}).$

But the percentage ratio of extension to original length $\Big\} = \dfrac{l}{L} \times 100 = \%\ e.$

∴ the Work done in inch-tons per cubic inch $= \dfrac{\%\ e}{300}\,(f_t + 2 f_{max.}).$

N.B.—The best practical method of arriving at the suitability of any given material for withstanding shocks and blows is by obtaining the work done in inch-tons per cubic inch of the test-piece.

EXAMPLE 1.—In a tensile test of a piece of flat wrought-iron bar, the following results were obtained :—

    (i.) Original dimensions of cross-section 2·010 inches by 0·505 inch.
    (ii.) Final dimensions of cross-section at point of fracture 1·520 inch by 0·410 inch.
    (iii.) Gross load at limit of elasticity 34,500 lbs.
    (iv.) Gross load at fracture 53,100 lbs.
    (v.) Total extension on length of 10 inches = 1·46 inch.
    (vi.) Extension on 10″ length under a gross load of 20,000 lbs. $= \frac{7}{1000}''$.

Find from the above data:—(a) The modulus of elasticity of the material; (b) the limit of elasticity and tenacity per square inch ; (c) the reduction of area per cent. ; (d) the approximate work done per cubic inch of the material in fracturing the bar. Would you consider this good material?

<div align="right">(I.C.E., Oct., 1901.[\]</div>

ANSWER.—

(a) Young's modulus of elasticity $\Big\}$ $E = \dfrac{P L}{A l}.$ (Equation IV., Lecture I., Vol. II.)

   ,,   ,, $E = \dfrac{20.000 \times 10}{2 \cdot 01 \times \cdot 505 \times \frac{7}{1000}} = \dfrac{200,000,000}{7 \cdot 105}$

   ,,   ,, $E = $ **28,143,000 lbs. per square inch.**

(b) Stress at elastic limit in lbs. per square inch $\Big\}$ $= \dfrac{\text{load at elastic limit}}{\text{original area}} = \dfrac{34,500}{1 \cdot 015}$

   ,,   ,, $= $ **34,000 lbs. per square inch.**

(c) Percentage contraction of area $\Big\}$ $= \dfrac{\text{original area} - \text{final area}}{\text{original area}} \times 100$

   ,,   ,, $= \dfrac{(2 \cdot 01 \times \cdot 505) - (1 \cdot 52 \times \cdot 41)}{2 \cdot 01 \times \cdot 505} \times 100$

   ,,   ,, $= \dfrac{(1 \cdot 015 - \cdot 6232)100}{1 \cdot 015} = \dfrac{39 \cdot 18}{1 \cdot 015} = $ **38·6 per cent.**

(d) Work done per cubic inch $\Big\}$ $= \dfrac{\text{work done in stretching the test-piece}}{\text{volume of the test-piece}}$

   ,,   ,, $= \dfrac{\left\{\begin{array}{l}\text{mean load between starting point and} \\ \text{point of fracture} \times \text{distance moved} \\ \text{through}\end{array}\right\}}{\left\{\begin{array}{l}\text{area in square inches} \times \text{length of test-} \\ \text{piece in inches}\end{array}\right\}}$

   ,,   ,, $= \dfrac{30,525 \times 1 \cdot 46}{10 \cdot 15} = $ **4,390 inch-lbs.**

(e) Percentage elongation on 10 inches $\Big\}$ $= \dfrac{\text{final length} - \text{original length}}{\text{original length}} \times 100$

   ,,   ,, $= \dfrac{1 \cdot 46 \times 100}{10} = $ **14·6 per cent.**

Influence of Time on Stress and Strain, as shown by the Diagram.—Prof. Barr says, another advantage in wire testing is, that any particular condition may be varied. The wire can be tested quickly or slowly, and where a considerable difference in the stress-strain diagrams is shown, it will be known to be due to the method of testing. Besides that, the material may be given rests more conveniently in experiments on testing wires than could be done with specimens in a large machine. The results obtained by varying the rate of application of the stress to the specimen as given by the stress-strain diagrams in Fig. 26, are exceedingly
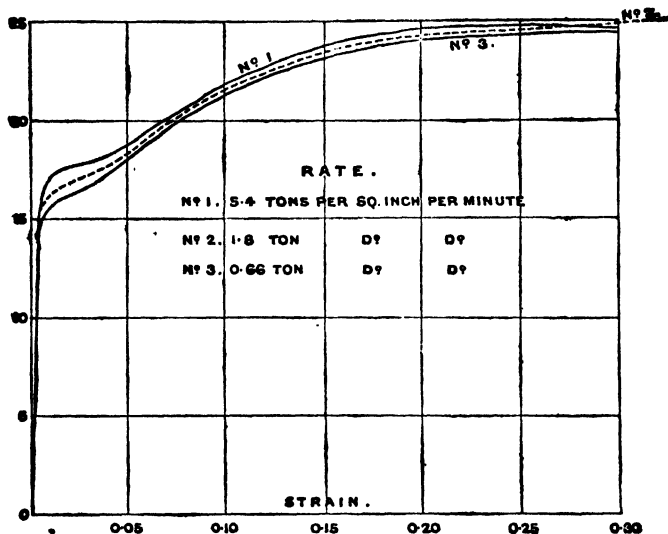
FIG. 26.—STRESS-STRAIN DIAGRAMS OF A TEST-PIECE OF CHARCOAL IRON UNDER THREE CONDITIONS OF TESTING (see Fig. 2, Lecture V., Vol. II.).

interesting, because it is certain, that almost the whole of the measurable difference of results on the diagram is due to the different treatment, and not to any difference in the specimens themselves. The arrangement shown in Fig. 2, is well designed for accurately obtaining the required variations in the rate of application of stress within a wide range.

Torsion Tests.—*Autographic Diagram taken with a Flat Disc Recorder.*[*]—Fig. 27 shows in a graphic way the behaviour of an "acid steel" specimen, ¾ inch in diameter and 6 inches long, subjected to a maximum torsion or twisting moment of 4,840 inch-lbs. The specimen

---

[*] The originals of both of these torsion diagrams were kindly taken by Messrs. Buckton & Co. and presented to the Author for this book. They were then reproduced to a convenient scale by his publishers.

was twisted through an angle of 1,202°. The series of concentric circles represent the increase of stress, while the radial divisions show the amount of torsion. A complete turn of the test-piece is represented by 360° or 2π radians. It will be seen, that the characteristic behaviour of the specimen is shown to be a rapid increase of stress for minute increments of torsion, until the elastic stress is attained. It is thereafter followed by a greater and greater amount of twisting compared to the increase of stress. Hence, in the case of tough and very ductile materials, there is during the last stage of the experiment a very large proportion of twist with very little increase of stress; or, as illustrated by Fig. 27, the curve becomes almost concentric during the later stages, instead of maintaining the spiral form as in the earlier stages. The arrowhead at the very end of the curve indicates that the specimen has broken.
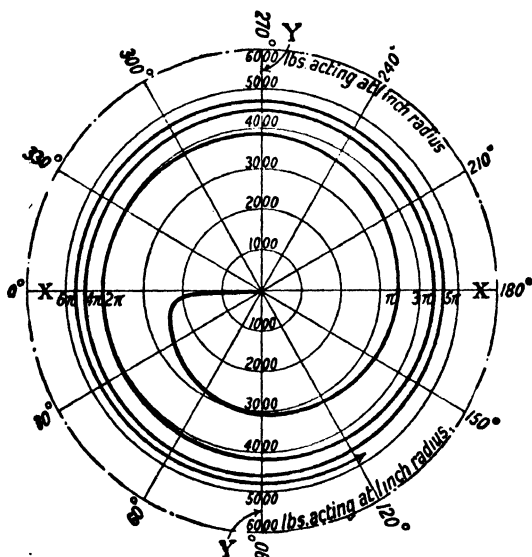
FIG. 27.—SPIRAL DIAGRAM OF TORSION TEST TAKEN WITH WICKSTEED'S FLAT DISC RECORDER. Made by Buckton & Co., Leeds.

If the curve in Fig. 27 be developed on a flat surface we get a somewhat similar diagram to that shown by Fig. 28. Here the ordinates represent twisting moments and the abscissæ angles of twist or torsion.

Autographic Diagram taken with a Drum Recorder.—Fig. 28 is a copy of the diagram taken by a drum recorder for a "basic steel" specimen 2 inches in diameter and 4 inches long between the collars. The maximum twisting moment was 103,290 inch-lbs., and the angle through which the specimen twisted was 450°, or 2·5 π.

To find the Work Done per Cubic Inch by aid of an Autographic Diagram, when subjecting a Test-Piece to a Torsion Test.—Take the case of Fig. 28, which represents a stress-
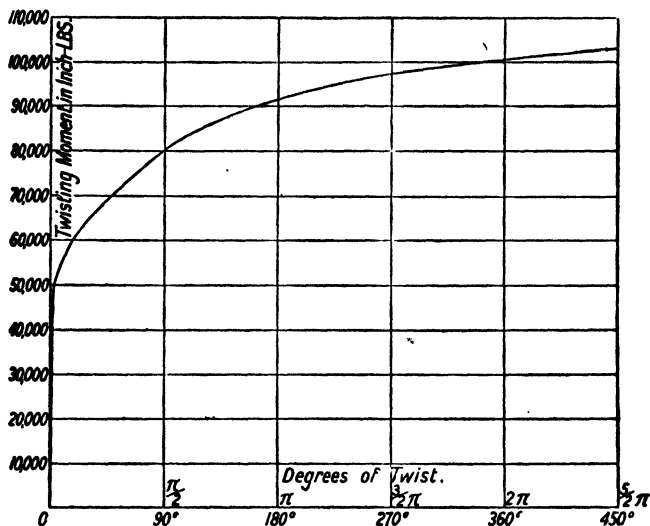
FIG. 28.—DIAGRAM OF TORSION TEST TAKEN WITH A DRUM RECORDER.
Made by Buckton & Co., Leeds.

strain diagram in which the ordinates represent twisting moments and the abscissæ angles of twist.   We get—

The *vertical scale* for twisting moments $\Big\}$ = 43,500 inch-lbs. per inch.

The *horizontal scale* for the angle of twist is taken as—
            1 inch = ·87π.

The *area* of diagram is approximately $\Big\}$ = 3·75 square inches.

Hence, Work done $= \dfrac{\text{area of diagram} \times 43{,}500 \times \cdot 87\pi}{2{,}240} = \dfrac{3\cdot 75 \times 43{,}500 \times \cdot 87 \times 22}{7 \times 2{,}240}$

           „   „   = 199·12 inch-tons.

And, Work done per cubic inch $\Big\} = \dfrac{\text{Work done in inch-tons}}{\text{Volume of bar in cubic inches}} = \dfrac{199\cdot 12}{3\cdot 14 \times 4}$

      „   „   = 15·8 inch-tons per cubic inch.

Searle's Apparatus for Measuring the Extension of a Wire and for Determining Young's Modulus of Elasticity.*—In studying the effects of longitudinal stress upon the length of a wire some accurate means is required for measuring the elongation of the test-piece. The simplest method of magnifying these effects, consists in using a wire of considerable length, which for convenience is hung from a fixed support. The extension is produced by hanging weights to the lower end of the wire. Two chief sources of error must be avoided at the outset. These errors arise from the yielding of the support and the change in length of the wire due to a rise of temperature. Both errors are practically eliminated if, instead of finding the displacement of the lower end of the wire relative to a fixed mark, observation be made of the displacement of the end of the wire relative to the lower end of a second wire of the same material, hanging from the same support, stretched by a constant weight and thus serving as a standard of comparison. It is here assumed, that the coefficients of expansion with temperature of the two wires are identical in spite of their differences in stress.
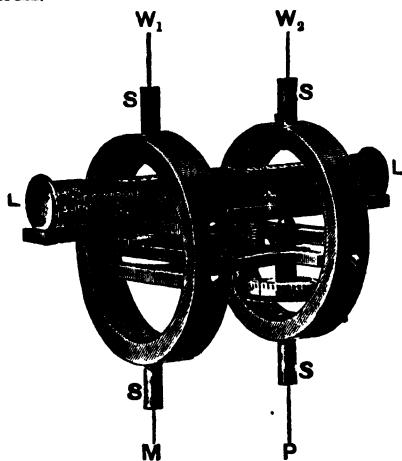
FIG. 29.—SEARLE'S APPARATUS FOR MEASURING THE EXTENSION AND DETERMINING YOUNG'S MODULUS FOR WIRES.

Description of Apparatus.—The apparatus shown by Fig. 29 furnishes a very sensitive means of measuring the relative displacement of the ends of the wires, and in addition to its sensitiveness, it possesses the advantage of giving direct readings. The two wires W_1, W_2 have their upper ends secured to a stout piece of metal bolted to a beam. From their lower ends hang two cylindrical brass frames supporting the two ends of a sensitive level L. One end of the level is pivoted to the frame at the left-hand side

---

* I am indebted to Mr. G. F. C. Searle, M.A., Demonstrator at the Cavendish Laboratory, Cambridge, for kindly supplying me with a copy of his paper on "Apparatus for Measuring the Extension of a Wire," see vol. v., part v., of the *Proc. of the Cambridge Philosophical Society,* from which the above article has been compiled.

of Fig. 29 by pivots; whilst the other end of the level rests upon the end of a vertical micrometer screw, working in a nut attached to the other frame at the right-hand side of the figure. The central link connecting the two frames prevents them from twisting, relative to each other about their vertical axes, but allows a free vertical relative motion. When this link is horizontal the two wires are parallel to each other. From the lower ends of the frames hang a mass M and a pan P. The weights of M and P are sufficient to ensure that the wires are straight. The connections between the two wires W_1, W_2 and the frames are made by the swivels S, into which the ends of the wires are soldered. These swivels enable the wires to be torsionless and thus ensure that the two wires hang in a vertical plane. Two other swivels S, connect M and P to the frames. The head of the vertical micrometer screw is divided into a number of equal divisions around its circumference, and a scale is engraved on the side of the frame at the right-hand side, which scale serves to determine the number of complete revolutions made by the screw.

Method of Using the Apparatus.—Suppose the screw has been adjusted, so that one end of the bubble of the level is at its fiducial mark. If a weight be placed in the pan P, the wire W_2 is stretched and the bubble moves towards the left-hand end of the level. The bubble is then brought back to its fiducial mark by turning the micrometer screw, so as to raise the right-hand end of the level resting upon it. The distance through which the screw is moved is clearly equal to the increase of length of the wire W_2, and is determined at once by the difference of the reading of the screw in the two positions. The level is sensitive enough to enable the screw to be adjusted to $\frac{1}{4}$ of a division on its graduated head.

With this apparatus it is easy to investigate the deviations from Hooke's Law for copper wires.

Prof. Ewing's Extensometer for Measuring the Elastic Extension and Young's Modulus of Elasticity of Specimens of Metal under Tensile Tests.

—This instrument can be quickly applied to any test-piece, and *no part of it has to be touched whilst the test is being made.*

Two clips B_1 and C are attached to the test-piece A by the points of two set-screws. The clip B_1 has a projection B_2 ending in a rounded point P, which engages with a conical hole in C. When the bar extends, this rounded point serves as a fulcrum for the clip C, and hence a point Q, equally distant on the other side, moves, relatively to B_1, through a distance equal to twice the extension. This distance is measured by means of a microscope attached to B_1. The microscope forms a prolongation of B_1, and the motion of Q is brought into the field of view by means of a hanging rod R. The rod R is free to slide on a guide in B_1, and carries a mark on which the microscope is sighted. The displacement is read by means of a micrometer scale in the eye-piece of the microscope. The pieces B_1 and B_2 are jointed to one another in such a way that the bar may twist a little, as it is sometimes liable to do during a test, without affecting the engagement of P with C. This also obviates any need of absolute parallelism in the axes of attachment of the two clips. But the joint between B_1 and B_2 forms a rigid connection so far as angular movement in the plane of the paper is concerned. This feature is essential to the action of the instrument, for it is only then that P serves as a fixed fulcrum in the tilting of A, by extension on the part of the specimen.

The figure is an illustration of the usual form of this instrument. The clips B_1 and C are set at 8 inches (200 mm.) apart.

Each division on the microscope scale corresponds to $\frac{1}{1000}$ inch of exten-

sion, so that, by estimation of tenths of a division, readings to $\frac{1}{15000}$ inch may be taken.

The screw L further serves to bring the sighted mark to a convenient point on the micrometer scale, and also to bring the mark back if the strain is so large as to carry it out of the field of view.   In dealing with elastic strains there is no need for this, as the range of the scale itself is sufficient to include them, but it is useful when observations are being made on the behaviour of metals when the elastic limit is passed.

To facilitate the application of the extensometer to any rod, a clamping bar is added, by which the clips B_1 and C are held at the right distance apart with the axes of their set-screws parallel, while they are being secured to the test-piece.   Such a clamping bar is especially convenient when the strain has been carried beyond the elastic limit, and it is desired to reset immediately, the clips to the standard distance apart, after the length between them has materially changed by extension of the specimen. The clamping bar must, of course, be removed before a test begins.
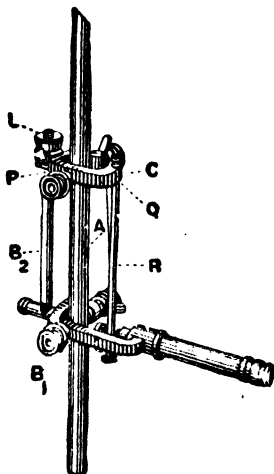
Fig. 30—Prof. Ewing's Extensometer.*

An apparatus can also be made for marking off on the specimen the 8-inch length to which the extensometer is applied when in use.   To use this latter apparatus, the set-screws are turned back so that the test-piece rests in the V-grooved blocks, the upper screws are then tightened upon the test-piece, and the lower screws are also tightened sufficiently to make their points indent the bar.

* I am indebted to Prof. J. A. Ewing, F.R.S., &c., and to "The Syndics of The Cambridge University Press" for their kind permission to reproduce in this lecture the following 7 figures :—31, 32, 34, and 36 to 39.   Those devised by Prof. Ewing are specially indicated with his name, and all are made by The Cambridge Scientific Instrument Co., Limited, from whom I received the other 3 blocks for figures 17, 29, and 35.   Most of these figures are illustrated and described in Prof. Ewing's treatise on "The Strength of Materials," as published by The Cambridge University Press.

Testing Machine with Extensometer Attached for Measuring the Elastic Extension of Rods.—Extensometers are commonly used in conjunction with testing machines. It is also necessary to note, that for many experiments on elasticity, the testing machine is not essential, because so long as elastic strains are being dealt with there is no need of hydraulic or other gearing to take up the stretch, and an ordinary lever may be all that is required for applying the load. The figure below illustrates a laboratory apparatus for measuring Young's modulus in rods of various metals. By placing weights on the vertical rod attached to the right-hand end of the lever, loads up to 1 ton can be applied to the test-piece, which may have a diameter of $\frac{1}{4}$ to $\frac{3}{8}$ inch. An extensometer is attached to the test-piece on the left-hand side of the lever, to measure the elongation produced in it.
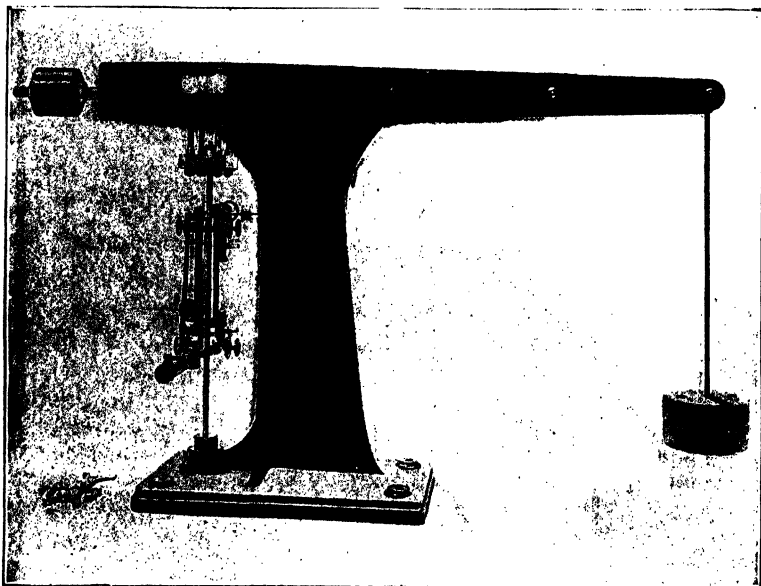
Fig. 31.—TESTING MACHINE WITH EWING'S EXTENSOMETER ATTACHED
FOR MEASURING THE ELASTIC EXTENSION OF RODS.

Measurement of Young's Modulus in Wires.—When long pieces of wire are available as test-pieces, then the elastic extension can be obtained by direct measurement by means of a scale and vernier. The usual plan is to hang the two wires side by side, fastening both to the same support and then attach the scale to one wire and the vernier to the other. The advantage of using the second wire to carry the scale is, that any yielding of the support or any change of temperature, such as might take place during the test, affects both wires equally. The wire with the scale attached to

It is kept taut by a load which is not varied during the test.   The other wire is first loaded with weights until it is straight, then additional weights are applied to produce the elastic extension.   This extension is measured by noting the movement of the vernier along the scale.   It is found, that with iron or steel wires 20 feet long, the extension is nearly $\frac{1}{16}$ inch for each ton per square inch of load.   As the load may generally
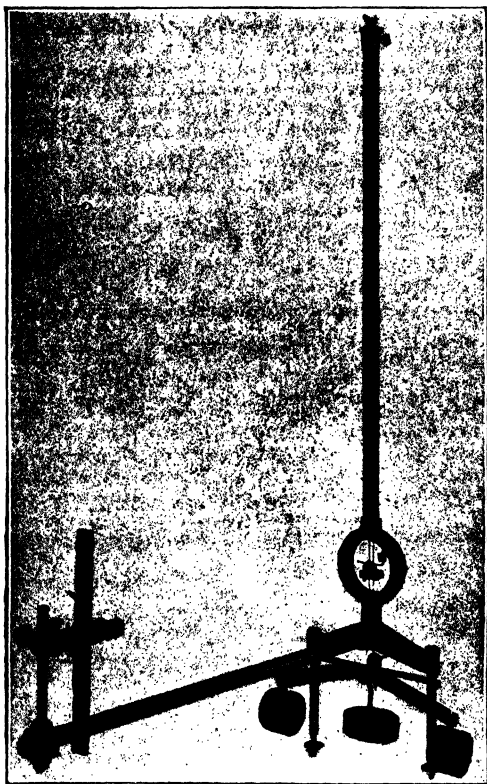
FIG. 32.—INSTRUMENT FOR MEASURING THE ELASTIC EXTENSION
OF WIRES.

be raised to 10 tons per square inch, and sometimes more, without passing the elastic limit, it will be seen, that the movement of the vernier will be sufficient to give approximately accurate values of the modulus of the material.

When short pieces of wire are only available, then some means of magnifying the relative displacement will be required.   One very good plan is to clamp two little blocks to the two wires.   These two blocks

2   13

serve as platforms, on which is placed a small tripod carrying a mirror. Two legs of the tripod are supported in a hole and a slot respectively on one of the blocks, while the third leg rests on a plane horizontal surface on the other block.* If one wire is stretched it causes the mirror to tilt, and the amount of this tilting is measured by means of the fixed telescope and scale shown at the left-hand side of the above figure.

It will be noticed, that the two wires hang inside the vertical tubular stem from the clamp and butterfly nut at the top. A cross-bar is attached to the bottom end of one of the wires which carries a constant load, while the variable load is applied to the other wire. The reading telescope with its attached scale is supported by part of the framework of the apparatus.

It is important to note, when calculating the extension of the test-piece from the scale readings, that the angle through which the reflected ray turns is twice the angle through which the mirror is tilted.

Thus, if w = The effective width of the tripod carrying the mirror—*i.e.*, the distance from its back leg to the line joining the two front legs.

n = Number of divisions on the scale by which the telescope reading has changed when a load is applied.

d = Distance between the mirror and the scale expressed in scale divisions.

Then, the small angle turned through by the ray = $\dfrac{n}{d}$.

And, the angle through which the mirror is tilted = $\dfrac{\delta l}{w}$, where δl is the extension of the wire.

∴ $\delta l = \dfrac{w n}{2 d}$ and is calculated in the same units as are used in measuring the effective width w of the tripod.

Measuring Young's Modulus by Bauschinger's Extensometer.

—The small strains which take place in a tensile test of a non-plastic material, and those strains which occur during the early part of the test in any kind of material, require some form of instrument
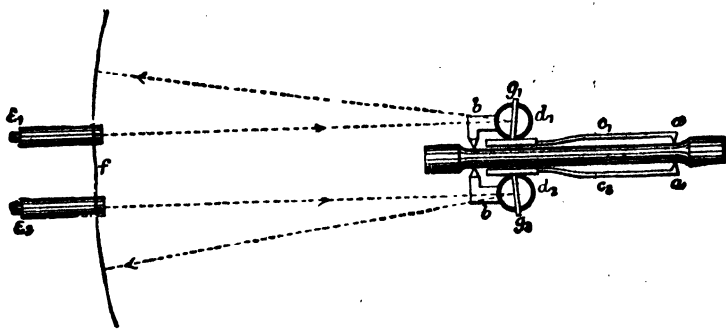
FIG. 33.—PROF. BAUSCHINGER'S OPTICAL APPARATUS FOR THE MEASUREMENT OF STRAINS.

* This automatic adjustment was first devised by Lord Kelvin in connection with the levelling of his quadrant electrometer.

capable of indicating delicate measurements. In measuring the elastic modulus and in determining the true elastic limit, it is necessary to be able to compare the fractional parts of the stretching of the test-piece, which are produced by each successive increase of load. Consequently, in order to obtain accurate measurements of the modulus of any material, there is a great advantage in being able to read the measurements to, say, $\frac{1}{1000}$ part of an inch, when we consider that the whole amount of elastic stretching in a test-piece of, say, wrought iron is only about $\frac{1}{1000}$ part of the length under observation. As the measurements taken between two marks on each side of the test-piece are so much affected by bending (due to inequality of the distribution of the stress), it is absolutely necessary, either to measure the extensions on opposite sides of the test-piece and take the mean, or to measure the displacements on each side of the test-bar by aid of two mirrors, &c., attached in such a manner as to indicate the strains on each side of the test-piece, as shown by the accompanying figure.

   Extensometer Applied to Measure the Elastic Compression of Short Blocks.—The illustration (Fig. 34) shows Prof. Ewing's extenso-meter as adapted by him to measure the elastic compression of short blocks
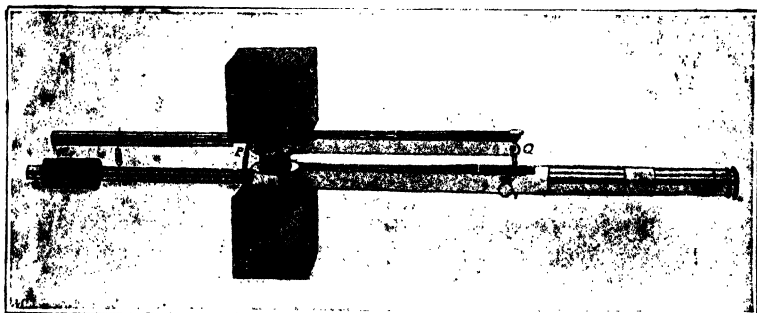
FIG. 34.—PROF. EWING'S EXTENSOMETER AS APPLIED TO MEASURE THE ELASTIC COMPRESSION OF SHORT BLOCKS.

of any material. It will be seen, that the length of test-piece which is to be tested between the clips is only $1\frac{1}{2}$ inches. The strain of the test-piece is multiplied ten times by a mechanical arrangement. This result is obtained by extending the clips to the right, and making the distance of Q from the axis nine times that of P. The prolongations of the clips to the left of the figure are added, to counterpoise the weight of the clips and microscope, &c., on the right-hand side, so that the force with which the point P presses against its bearing or socket may be vertical. The motion of the end Q of the lever P Q is transferred to the field of the microscope by means of a vertical hanging piece jointed to the lever at Q and carries the mark on which sights are taken. With this instrument no calibrating screw is required, whilst the object to be sighted by the microscope is a small piece of glass on which two horizontal lines are engraved at a distance of $\frac{1}{10}$ inch apart. The length of the microscope is arranged by adjustment to make these lines include 500 units of the eye-piece scale. Consequently, each unit corresponds to a displacement of the glass plate through $\frac{1}{5000}$ inch, or to an extension of the test-piece of $\frac{1}{50000}$ inch.

Apparatus for Determining the Deflection of Beams and Cantilevers.—This instrument consists of two cast-iron brackets which can be fixed to a wall. An iron bed having two movable supports with knife edges in order to test the rod when placed as a beam; also a cast-iron block for fixing the end of the test-piece when being tested as a cantilever, and shown by the figure. The deflection or lowering of the
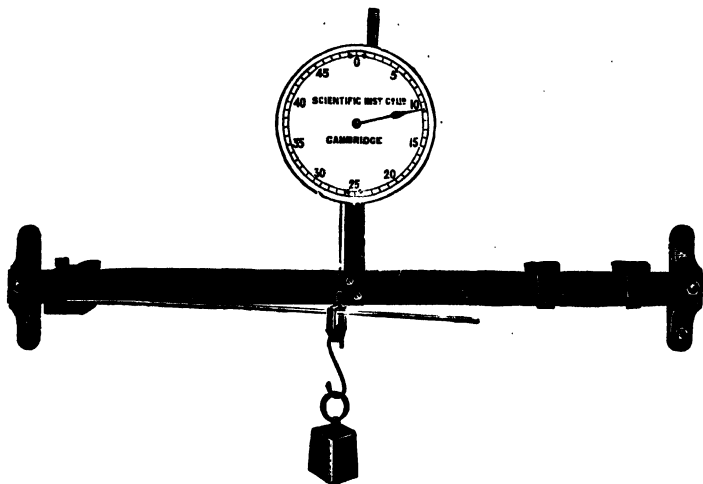
FIG. 35.—APPARATUS FOR DETERMINING THE DEFLECTION OF BEAMS AND CANTILEVERS.

middle point of the beam or the end of the cantilever may be small, so that this deflection is magnified by passing a cord from the top of the stirrup round the little axle which carries the pointer. The pointer will show on the dial the magnification of the deflection. The weights are attached by a carrier to the lower end of the stirrup.

Measurement of Young's Modulus of Elasticity by Deflection of Beams.—By noting the deflections of a loaded bar of the material, we have another ready method for finding Young's modulus for the material of the test-piece. The bar is supported as a beam on fixed knife-edged supports, or clamped at one end and free at the other. When the test-piece is long and sufficiently flexible to bend easily, then the deflection is measured by a fixed scale behind the test-piece, with a piece of mirror fixed alongside the scale. The readings may be directly taken by the experimenter bringing his eye to the level of the test-piece until the top edge of the test-piece just covers its reflection in the mirror, and then sighting this position of the edge upon the scale at its side. It is, however, found, that when dealing with less flexible test-pieces, an apparatus like that illustrated by Fig. 36, is found useful. In this arrangement, the knife-edged supports are clamped on a stiff bed similar to that of a lathe bed, and can be adjusted to suit any length of test-piece. The deflection of the

test-piece is measured by sighting a fine, accurately-divided glass scale, through a low reading microscope. This divided glass scale is clamped to the middle of the test-piece, as seen from the figure. A little jockey mirror can be set astride the test-piece at any point for the purpose of observing the angle of slope at that point. This mirror is shown at the right-hand support in the illustration. When the test-piece is loaded, the tilting of this mirror can be observed from a distance by means of another reading telescope and scale. It is sometimes more convenient, instead of loading the test-piece in the centre, to place two equal loads at the extremities, which are arranged to project by equal distances beyond the two supports. The great advantage of this latter method of loading is, that the middle portion of the test-piece is subjected to uniform bending and to no other kind of stress.
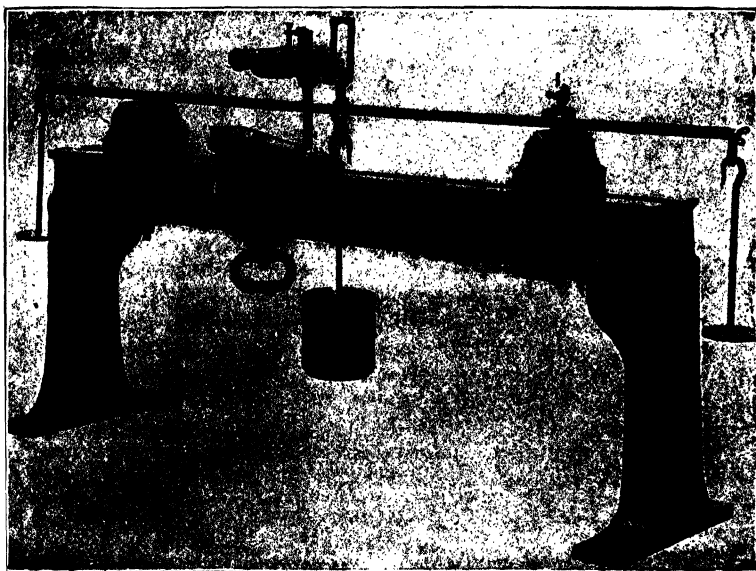
FIG. 36.—APPARATUS FOR MEASURING YOUNG'S MODULUS OF ELASTICITY BY DEFLECTION OF BEAMS.

Let L = The whole length of test-piece.

,, x = Half the distance between the supports.

,, $\dfrac{L-2x}{2}$ = $\begin{cases} \text{The distance by which the test-piece projects beyond} \\ \text{each support.} \end{cases}$
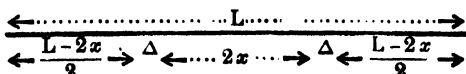
Then, as was shown in Lecture III., if a load W is applied at the centre of a beam, the deflection—

$$\Delta_1 = \frac{W L^3}{48 E I} = \frac{W x^3}{6 E I}. \qquad \therefore \quad E = \frac{W x^3}{6 \Delta_1 I}.$$

Where I is the moment of inertia of the section of the rod or beam about a horizontal central axis.

Again, if a load W be applied at each extremity of the test-piece, the upward deflection at the centre is given by the equation—

$$\Delta_2 = \frac{W x^2 (L - 2 x)}{4 E I}. \qquad \therefore \quad E = \frac{W x^2 (L - 2 x)}{4 \Delta_2 I}.$$

Young's Modulus by Deflection of Cantilevers.—The following Fig. 37 shows an arrangement for observing the deflection of the test-piece when used as a cantilever or beam fixed at one end and free at the other.
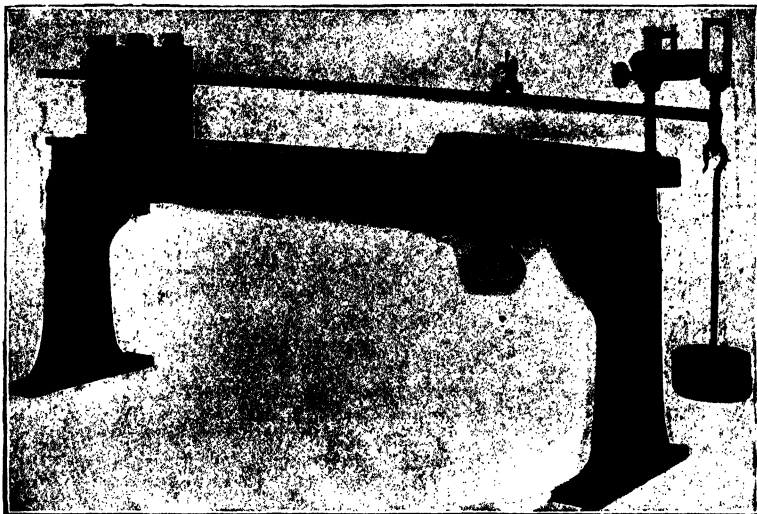
FIG. 37.—APPARATUS FOR MEASURING YOUNG'S MODULUS OF ELASTICITY BY DEFLECTION OF CANTILEVERS.

Assuming the load W to be applied at the free end, and, that L denotes the whole length of the test-piece from the clamp to the free end, we get the deflection, $\Delta = W L^3/3 E I$.

There is one great objection to this latter method of determining E, on account of the difficulty of maintaining the fixed end of the test-piece horizontal by means of the clamp. In this experiment, provision is made for observing the deflection at the various points along the test-piece, in order to plot the curve which a beam of uniform section will assume under a given load or system of loads. The slope of the cantilever may be determined from point to point along the test-piece ·by means of the jockey mirror, as previously described for slope of beams.

Measurement of the Modulus of Rigidity by the Torsion of Wires.*—The accompanying illustration shows a self-contained apparatus

for performing experiments on the torsion of wires. The wire to be tested is suspended in the vertical axis of a tubular stem. It carries a cylindrical weight, round which two cords pass and are led away over pulleys on both sides of the framework to hangers. Equal weights are placed upon the hangers, and the wire is consequently twisted by a pure couple. The angle of twist is read by observing the displacement of a pointer on a fixed circular scale. This pointer is attached to the upper end of the cylindrical weight.

When a rod is twisted every part of it is in a state of shear.

Let θ = Angle of twist expressed in circular measure.

  ,, l = Length of wire.

  ,, d = Diameter of wire.

  ,,   C = Modulus of rigidity.

  ,,   T M = Twisting moment.

Then, within the elastic limit, we get from Lecture IV., p. 106—

$$\theta = \frac{32\, l\, T\, M}{\pi\, d^4\, C}.$$

Or, $$C = \frac{32\, l\, T\, M}{\pi\, d^4\, \theta}.$$

It is found convenient, in applying this method to measure C in rods of moderate diameter, to find θ by using two long pointers clamped on the wire near its ends. The distant ends of the pointers pass over fixed scales. Hence, the difference in the two scale readings measures the angle of twist on the length l of the wire between the positions of the two pointers. Should the diameter of the wire be so great as to make the angle of twist too small to be measured in
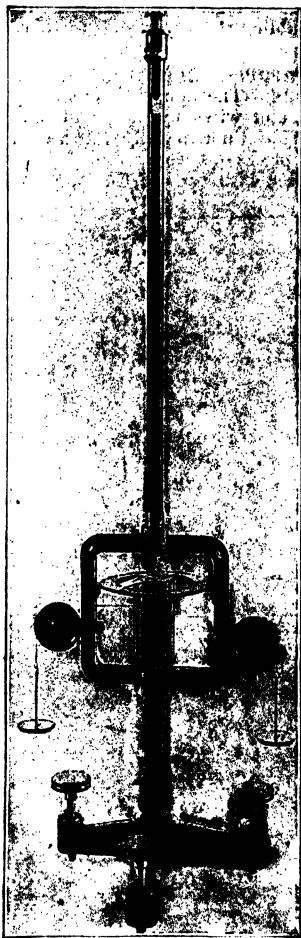
Fig. 38.—APPARATUS FOR MEASURING THE MODULUS OF RIGIDITY BY THE TORSION OF WIRES.

* See *Proc. Inst. C. E.*, vol. cli., 1903, for abstract of paper on "Iron and Steel under Torsional and Combined Stresses," by Prof. E. G. Coker, M.A., D.Sc.

this way, then a pair of mirrors are clamped on the wire and made to face sideways, and are used along with a reading telescope and scale for each. The advautage of an optical pointer over a mechanical pointer is, that of doubling the angle, and, further, it can readily be made longer.

Measurement of the Modulus of Rigidity by Torsional Oscillations.

The figure shows a complete apparatus for experiments on the torsion of wires by means of Maxwell's needle. It consists of a brass tube placed horizontally, into which there can be slipped four other equal short pieces of tube, and thus fill up its whole length. Two short pieces are filled with lead and the other two pieces of tube are empty. By placing, first, the two empty short tubes in the interior with the two loaded pieces of tube to the outer ends of the long tube, and second, reversing this order by placing the loaded pieces in the centre and empty ones at each end, you can obtain two different values of the moment of inertia of the system.

Let I_1 = Moment of inertia of system in the first arrangement of the short tubes.

" I_2 = Moment of inertia of system in the second arrangement of the short tubes.

" x = Half the length of the long tube.

" m_1 = Mass of each of the two short tubes which are filled with lead.

" m_2 = Mass of each of the empty tubes.

" t_1, t_2 = The observed periods of oscillation in the two arrangements respectively.

FIG. 39.—APPARATUS FOR EXPERIMENTS ON THE TORSION OF WIRES BY MEANS OF MAXWELL'S NEEDLE.

Then the system is changed by shifting two masses, each equal to $m_1 - m_2$, so that the distance of the centre of gravity of each mass from the axis changes from $\frac{3}{4}x$ to $\frac{1}{4}x$.

Hence to express the change in the moment of inertia—

$$I_1 - I_2 = 2(m_1 - m_2)(\tfrac{9}{16}x^2 - \tfrac{1}{16}x^2) = (m_1 - m_2)x^2.$$

But,
$$\frac{t_1^2}{t_2^2} = \frac{I_1}{I_2}, \text{ or } \frac{t_1^2}{t_1^2 - t_2^2} = \frac{I_1}{I_1 - I_2} = \frac{I_1}{(m_1 - m_2) x^2}$$

Or,
$$\frac{I_1}{t_1^2} = \frac{(m_1 - m_2) x^2}{t_1^2 - t_2^2}.$$

Also, the modulus of rigidity C can be got by the following formula :—

$$C = \frac{128 \, \pi \, l \, I_1}{g \, d^4 \, t_1^2} \text{ (see Lecture IV., p. 105)}.$$

Where t is the period of time taken to make each complete oscillation ; l is the length and d is the diameter of rod, and g is the factor for converting the twisting moment per unit of angle, or the constant ratio of the twisting moment T M to the angle θ into kinetic units.

Consequently, we obtain, without any calculations for moment of inertia I, the equation for the modulus of rigidity

$$C = \frac{128 \, \pi \, l}{g \, d^4} \left[\frac{(m_1 - m_2) x^2}{t_1^2 - t_2^2} \right].$$

Several other forms of extensometer, differing in detail construction from those already described, are now on the market.

LECTURE VII.—QUESTIONS.

1. Select any two of the five autographic apparatus for recording stress-strain diagrams which you consider to best fulfil the objects to be attained. Then sketch and explain their construction, use, and application.

2. Explain, by aid of a sketch of a complete stress-strain diagram, the several points of chief interest which occur in a rod of soft iron or mild steel, from the time of applying the stress until the rod breaks.

3. Explain, by aid of a sketch, how Prof. Kennedy automatically obtained elastic strain diagrams.

4. What is meant by Plasticity? How would you obtain a measure of the work done per cubic inch by aid of an autographic diagram when subjecting a test-piece to a tensile stress? Explain, also, Prof. Kennedy's approximate method and give the formula, with an example.

5. Explain the influence of time on the stress and strain of a metal as exhibited by a stress-strain diagram.

6. Explain, by aid of sketches, how you would make and take a torsional stress-strain diagram. Draw and explain torsion diagrams taken by disc and drum recorders.

7. Of the several methods given in this lecture for measuring the extension of wires and of determining Young's modulus of elasticity, select any one and describe it fully, by aid of sketches, with index to parts, &c.

8. Sketch and explain Prof. Ewing's extensometer for wires and rods.

9. Explain clearly, by aid of sketches, Bauschinger's extensometer for measuring Young's modulus of elasticity.

10. Sketch clearly and explain concisely Ewing's extensometer for measuring the elastic compression of short blocks.

11. Explain by sketches any apparatus for measuring Young's modulus of elasticity by deflection of beams and cantilevers. Work out the formulæ for the deflection and for Young's modulus of elasticity.

12. Sketch and explain any apparatus for measuring the modulus of rigidity by the torsion of wires.

13. Show how the mere torsional oscillations of a suspended weighted wire may be used to ascertain the modulus of rigidity.

14. Suppose the vertical loads and supporting forces of a horizontal beam to be known, show how we find (1) the shearing force at the section, (2) the position of the neutral line, (3) the compressive stress at any part of the section, (4) the curvature of the beam, (5) the new shape of a portion of the section originally rectangular, its sides vertical and horizontal.

15. Describe carefully the behaviour of a mild-steel bar tested gradually in tension up to rupture, and sketch a stress-strain curve for it. The following data were obtained in such a test:—Original diameter of bar $1\frac{1}{4}$ inches, final diameter at point of fracture $\frac{14}{16}$ of an inch, total load when limit of elasticity was reached 20·8 tons, total load at fracture 36·7 tons, total extension at fracture (on a length of 10 inches) 2·23 inches, elongation (of 10-inch length) under a total load of 10 tons ·005 inch. Deduce from these data the following :—(a) The modulus of tensile elasticity. (b) The reduction of area per cent. (c) The elongation at fracture per cent. (d) The limit of elasticity in tons per square inch. (e) The maximum load in tons per square inch (both for original and final area).

16. A cylindrical steel pin is used to couple together two rods, and the joint is arranged in such a way that the pin is subjected to double shear. If the total tensile force tending to pull the joint asunder is 18¾ tons, what diameter would you make the pin? Choose for yourself the working shearing stress which can be permitted.

17. Answer only *one* of the following :—(a) Describe a laboratory experiment by which you could find E, Young's modulus of elasticity for an iron wire 10 feet long and 0·05 inch diameter. How would you secure the upper end of the wire? How apply the load? And how measure the elongation? How would you plot your results, and how deduce the value of E? About how much elongation would you expect for a load of 15 lbs.?

18. Answer only *one* of the following:—(a) Describe a laboratory experiment by which you could find E, Young's modulus of elasticity, for an iron wire, 10 feet long and 0·05 inch diameter. How would you secure the upper end of the wire? How apply the load? And how measure the elongation? How would you plot your results, and how deduce the value of E? About how much elongation would you expect for a load of 15 lbs.?
(b) Sketch an apparatus for determining the coefficient of sliding friction between two planed surfaces of oak. If you have made this or a similar experiment, describe the behaviour of the sliding piece and any troubles you may have had. State how you would conduct the experiment, so as to establish the principal facts concerning such friction.

19. In a tensile test of mild steel the original section was a rectangle of dimensions 1·96 inches by ·4 inch, and the breaking load was 25 tons. The extension at fracture, in a length of 8 inches, was 2·44 inches and the contracted dimensions were 1·41 inches by ·27 inch. Under a load of 14 tons (within the elastic limit) the extension in 8 inches was ·0107 inch. Calculate the stress at breaking point, the percentage extension, the percentage contraction of area, and the value of Young's modulus. Express an opinion whether the specimen is a good or a bad one.

LECTURE VII.—I.C.E. QUESTIONS.

1. Sketch neatly and describe an extensometer suitable for the determination of the Young's modulus of the material of a specimen 8 inches or 10 inches long, noting any prominent features in the design and the reason for their adoption. Define "elastic limit" and "yield point."

2. A square bar of isotropic material, stressed within the elastic limit, is loaded so that the stress intensity over each section is (1) uniform, (2) proportional to the distance from a line drawn through the centre parallel to one side, tension on one side and compression on the other side of the line. Calculate, in terms of the maximum stress and the Young's modulus, the energy stored per unit volume in both cases. *Ans.* (1) $\dfrac{f^2}{2\,\mathrm{E}}$; (2) $\dfrac{f^2}{6\,\mathrm{E}}$.

3. Exhibit, on a diagram drawn approximately to scale, the relation between the stress applied and the resultant strain in mild steel (suitable for bridge work) and annealed copper bars about 10 diameters long.

4. In a tensile test of a wrought-iron bar, 0·75 inch in diameter and 10 inches between the gauge points, the following results were obtained:—

Stress in lbs. per square inch,	8,000	16,000	24,000	32,000	38,000	40,000	44,000
Extension per cent.,	0·027	0·056	0·081	0·113	0·135	2·7	4·2

Stress in lbs. per square inch,	48,000,	52,000	55,000
Extension per cent.,	6·5	10·7	23·7

The yield point is 38,000 and the specimen broke at the maximum stress, 55,000. Plot the stress-strain diagram. Calculate the Young's modulus and find approximately the work done per cubic inch up to fracture.

5. What is meant by the "steelyard drop" or the "yield point"? At what load would you expect to find it in test-pieces of—(a) wrought iron, (b) 27-ton steel? *Ans.* (a) 13·; (b) 15 tons per square inch.

6. Describe briefly the observed effects of a direct compressive load applied to cubes or short cylinders of the following materials:—(a) Sandstone, (b) mild steel, (c) cast-iron, (d) timber placed with the grain on end.

7. Sketch the form of the stress-strain diagram, showing the plastic elongation of tough mild steel as commonly observed under direct tension.

8. Describe and illustrate some of the methods and appliances for the accurate measurement of elastic extensions, and show how Young's modulus is determined.

9. How is the strength of the material affected when rivet-holes are punched in thick or thin plates of mild steel, or in the foot of a Vignoles rail of harder steel?

10. Sketch approximately to scale stress-strain curves as drawn by an automatic recording apparatus for test bars of wrought iron, mild steel, and soft rolled brass. State what you would expect the load per square inch to be at the limit of elasticity and at fracture; also the maximum stress for each of the bars.

11. Describe any instrument with which you are aquainted for measuring the elastic tension in a test-bar.

12. Describe some form of "drop weight" or "impact" testing machine.

13 How would you prepare the following materials for compression tests :—A block of sandstone about 3 inches cube, a brick, a piece of cast iron 10 inches long and 1 inch square in section ?

14. Sketch an apparatus suitable for taking the necessary measurements for obtaining the modulus of elasticity of steel, and briefly describe the same.

15. When testing a piece of steel in an ordinary testing machine, explain clearly what happens to the test-piece up to the moment of fracture, and draw a stress-strain diagram.

16. Define "stress" and "strain." The following data were obtained from a tensile test on a mild steel bar :—Original dimensions : length, 8 inches ; area of cross-section, 0·62 square inch. Final dimensions : length, 10·2 inches ; area of cross-section, 0·34 square inch. Load at elastic limit, 13 tons ; maximum load, 19·6 tons. Mean extension per ton of load up to elastic limit, 0·001 inch. Determine the intensity of stress at elastic limit ; the intensity of the breaking stress ; the contraction of area per cent.; and the modulus of elasticity. *Ans.* 21 tons per square inch ; 57·6 tons per square inch ; 45 per cent. ; 12,900 tons per square inch.

17. Describe a testing machine for making a tensile test to destruction of a piece of mild steel. Sketch the stress-strain diagram you would expect to get, and mark on it the chief points to be noted, giving approximate values. What measurements are made of the test-piece itself, and explain with what objects they are taken ?

18. State the characteristics of the " plastic state " of a material. Describe the observed effects of direct compressive loading applied to cubes or short cylinders of (*a*) mild steel, (*b*) cast iron, (*c*) timber placed with the grain on end.

19. Draw roughly to scale the load-extension diagram of the tensile test of a steel bar 1 inch in diameter, having a maximum strength of 32 tons per square inch ; indicate the points in the diagram where changes in the structure occur.

20. State briefly what effects the following conditions have upon the testing of steel :—(*a*) The length of the test-piece, (*b*) the sectional area of the test-piece, (*c*) the shape of the test-piece, (*d*) the time occupied in making the test.

LECTURE VIII.

STRENGTH AND ELASTICITY OF COLUMNS.

Short Columns and the Effects of Imperfect Loading.[*]—If a cast-iron strut or short column be straight before loading, it does not

necessarily follow that the column will remain so when loaded; because one side of the column in cooling may become harder than the other, when naturally the softer side would yield the most. Or, the load may not be applied at the centre, due either to the ends of the column not being true and parallel, or the line of direction of the load acting to one side of the central line of the column.

If the total load P acts along the axis of the short column, the ratio of whose length to diameter is not greater than 3 to 1, then the column will fail by direct crushing. The intensity of compressive stress on the section will be—

$$f = \frac{P}{A},$$

where A = area of cross-section of column.

Columns of medium length fail partly by crushing and partly by bending, while very long columns fail entirely by bending.

Rankine's Rules for the Strength of Columns of Medium Length.—The late Professor Rankine gave the following rules for calculating the strength of struts and columns. These rules are expressed in terms of the least radius of gyration of the section, and they are of the greatest importance :—

Fig. 1. — Sketch of a Loaded Column when Hinged or Free at Both Ends.

[*] See *Proc. Inst. C.E.*, vol. cxxxiv., 1898, for paper on "The Distribution of Strain in a Flat Bar subjected to Unsymmetrical Stress," by William Ernest Dalby, M.A., B.Sc., Assoc.M.Inst.C.E. See *Engineering*, Oct. 14th, 1887, for article by Prof. R. H. Smith. See *Proc. Inst. C.E.*, vol. lxxxvi., 1885-86, for paper on "The Practical Strength of Columns, and of Braced Struts," by Thomas Claxton Fidler, M.Inst.C.E.

Case (1).— *When the Column is Hinged or Free at both Ends.*

Let O A = The original straight axis of the column. Taking this line
as axis of x, and the extremity O for origin.

" P = Breaking load of the column.

" A = Area of cross-section of the column.

" l = Length of the column.

" y = Co-ordinate of the centre of area of a normal cross-section,
and the deflection at abscissa x respectively.

" Δ = Maximum deflection of the middle of the column from the
vertical, or d in the plotted results of Fig. 4.

" $f_{o\,max.}$ = Maximum intensity of compressive stress.

" f_o = Mean compressive stress.

" f_b = Intensity of stress due to bending $= f_{o\,max} - f_o$.

" k = Least radius of gyration of the cross-section of the column.

" c = Coefficient whose value depends upon the material.

$$f_{o\,max.} = \frac{P}{A}. \qquad \qquad \text{(I.)}$$

We know, that $f_b = \dfrac{M\,y}{I}$ (see equation V., Lecture III., Vol. II.)

But maximum M = P Δ. $\therefore f_b = \dfrac{P\,\Delta\,y}{I}$.   (II.)

Let ρ be the radius of curvature of the bent axis of the column, then to
find the maximum deflection Δ, we get—

$$\left(\frac{l}{2}\right)^2 = \Delta\,(2\,\rho - \Delta) = 2\,\rho\,\Delta. \quad \therefore \Delta = \frac{l^2}{8\,\rho}. \qquad \text{(III.)}$$

But, $\dfrac{f_b}{y} = \dfrac{E}{\rho}$.   Consequently, from equation (III.)

$$\Delta = \frac{l^2}{8\,\rho} = \frac{l^2 f_b}{8\,E\,y} = c\,\frac{l^2}{y}, \text{ where } c \text{ is a constant} = \frac{f_b}{8\,E}.$$

Substituting this value in equation (II.), we obtain—

$$f_b = \frac{P\,\Delta\,y}{I} = \frac{P}{I}c\,l^2 = \frac{P}{A}\,c\,\frac{l^2}{k^2} = c f.\frac{l^2}{k^2},$$

where k is the radius of gyration, and $I = A\,k^2$ (see Lecture XII., Vol. I.)

Then the maximum intensity of stress in the column

$$f_{o\,max.} = f_o + f_b = f_o + c f_o\,\frac{l^2}{k^2} = f_o \left(1 + c\frac{l^2}{k^2}\right)$$

Or,   " $= \dfrac{P}{A}\left(1 + c\,\dfrac{l^2}{k^2}\right).$

If $f_{o\,max.}$ is the maximum stress allowed, and which must not be greater
than the elastic limit stress in compression of the material, then

$$\text{Breaking load} = P = \frac{f_{o\,max}\,A}{1 + c\,\dfrac{l^2}{k^2}}. \qquad \qquad \text{(IV.)}$$

Where the radius of gyration k of the section with respect to the axis
about which the resistance to bending is least—viz., the axis about which
I is least. The steady working load to be placed upon the column should

not be greater than n P, where $n = \frac{1}{4}$ to $\frac{1}{3}$ for wrought iron and steel, $\frac{1}{4}$ for cast iron, and $\frac{1}{10}$ for wood, whilst for live loads these values should be halved.

Case (2).—*When the Column or Strut has one End Fixed and the other End Rounded or Jointed.*—Then we must substitute $\frac{2}{3} l$ for l, and we get—

$$P = \frac{f_{o\ max.}\ A}{1 + \frac{4}{9}\frac{c\ l^2}{b^2}} \qquad . \qquad . \qquad . \qquad . \qquad \text{(V.)}$$

Case (3).—*When the Column or Strut has Fixed Ends.*—If both ends of the column are fixed, then the load which it will carry before bending is the same as for a strut of half the length hinged at the ends, and we must substitute $\frac{l}{2}$ for l.

$$\therefore \qquad P = \frac{f_{o\ max.}\ A}{1 + \frac{c\ l^2}{4\ b^2}} \qquad . \qquad . \qquad . \qquad . \qquad \text{(VI.)}$$

Case (1).   Case (2).   Case (3).
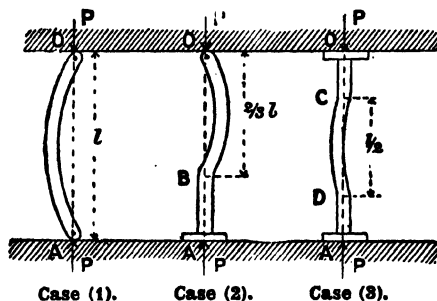
Fig. 2.—Sketch showing Columns with Hinged, Hinged and Fixed, and Fixed Ends respectively.

Gordon's Formula for Columns.

Taking the same notation as that given above for Rankine's formula

Let d = the greater dimension of the cross-section of column.
   ,, b = the least dimension of the cross-section of column.
   ,, $A = d\,b$ = cross-sectional area of column.

Maximum bending moment, $M = P\,\Delta$, but $\Delta \propto \frac{l^2}{b}$. $\therefore M \propto \frac{P\,l^2}{b}$.

Also, $f_b \propto \dfrac{M}{d\,b^2} \propto \dfrac{P\,l^2}{d\,b^3} \propto \dfrac{P\,l^2}{A\,b^2} = c\,f_o\,\dfrac{l^2}{b^2}$,   where c is a constant.

Hence, maximum intensity of stress,

$$f_{o\ max.} = f_c + f_b = f_o\left(1 + c\frac{l^2}{b^2}\right) = \frac{P}{A}\left(1 + c\frac{l^2}{b^2}\right). \qquad \therefore \quad P = \frac{f_{o\ max.}\ A}{1 + c\frac{l^2}{b^2}}.$$

This latter equation, which is known as Gordon's formula, is similar to Rankine's one, except that the least breadth of section b is taken instead

of k the least radius of gyration, and the value of the constant c modified accordingly. This formula will be true for circular and rectangular sections, since the least dimension of cross-section is a function of the diameter of the column, but for all other sections the least dimension of cross-section is not a simple function of the diameter, hence Rankine's formula is more general than Gordon's, and is to be preferred.

VALUES OF THE CONSTANTS IN RANKINE'S FORMULÆ.

Material.	$f_{o\ max.}$ in Tons per Square Inch.	c.
Cast iron, ...	35	$\frac{1}{1600}$
Wrought iron, ...	16	$\frac{1}{9000}$
Mild steel, ...	21	$\frac{1}{7500}$
Hard steel, ...	30	$\frac{1}{5000}$

VALUES OF THE CONSTANTS IN GORDON'S FORMULA.

Material.	Form of Section	Values of c.			$f_{o\ max.}$ in Tons per Square Inch.
		Ends Rounded or Pivoted.	Ends Fixed.	One End Fixed and the Other End Pivoted.	
Wrought iron,	L, T, H, channel, and hollow square, . .	$\frac{4}{900}$	$\frac{1}{900}$	$\frac{1}{500}$	19
	Hollow round, .	$\frac{1}{900}$	$\frac{1}{3600}$	$\frac{1}{2000}$	17
	Solid round, .	$\frac{1}{800}$	$\frac{1}{2250}$	$\frac{1}{1250}$	16
	Solid rectangular,	$\frac{1}{700}$	$\frac{1}{3000}$	$\frac{1}{1700}$	16
Cast iron, .	Hollow round, .	$\frac{1}{200}$	$\frac{1}{800}$	$\frac{1}{400}$	35
	Solid round, .	$\frac{1}{100}$	$\frac{1}{400}$	$\frac{1}{380}$	35
Mild steel, .	Hollow round, .	$\frac{1}{625}$	$\frac{1}{2500}$	$\frac{1}{1400}$	30
	Solid round, .	$\frac{1}{360}$	$\frac{1}{1400}$	$\frac{1}{780}$	30
	Solid rectangular,	$\frac{1}{330}$	$\frac{1}{2480}$	$\frac{1}{1350}$	30

EXAMPLE I.—State Gordon's formula for the strength of columns, and show how it is obtained. Apply it to find the breaking load for a cast-iron column 8 inches external diameter, $6\frac{1}{2}$ inches internal diameter, and 22 feet high. The column has flat ends. Take $f = 80,000$, and $a = \frac{1}{600}$.

ANSWER.—The first part is already answered in the text.

By Gordon's formula, $P = \dfrac{f_{c\ max.}\ A}{1 + c\dfrac{l^2}{b^2}}$.

Where, $c = a = \frac{1}{600}$; and $f_{c\ max.} = f = 80,000$ lbs.

Substituting values in the formula, we obtain—

$$P = \frac{80,000 \times (8^2 - 6\cdot5^2) \times \cdot7854}{1 + \dfrac{1}{600}\left(\dfrac{22 \times 12}{8}\right)^2} = \frac{80,000 \times 17}{2\cdot77} \text{ lbs.}$$

$$\therefore \qquad P = \frac{491,000}{2,240} = 220 \text{ tons.}$$

Long Columns.—In a long column, the ratio of its length to the least radius of gyration may be sufficient to cause the column to fail by lateral flexure rather than by direct crushing. Consequently, we have a compound stress somewhat similar to that which exists in the short columns with eccentric loading. The tendency to lateral flexure in long columns may be increased by the eccentricity of the loading caused by the direction of the line of action of the load not coinciding with the true axis of the column. The strength of long columns has been investigated mathematically by Euler and Rankine, and more recently by Professors R. H. Smith, Claxton Fidler, and others.

Proof of Euler's Formula for Long Columns (see Fig. 1).

1st. *When the Column is Hinged or Rounded at the Ends.*—Let O B A be the bent axis of the column. Take the origin at O, and the vertical line O A as the axis of x. Let y be the deflection at the intermediate point c of the column, $\rho =$ radius of curvature, and M = bending moment at $x y$.

Then, $\dfrac{1}{\rho} = \dfrac{M}{E\,I} = \dfrac{P\,y}{E\,I}$; and $\dfrac{1}{\rho} = -\dfrac{d^2 y}{d x^2}$.

Or, $\dfrac{P\,y}{E\,I} = -\dfrac{d^2 y}{d x^2}$.

This negative sign is used, because, if we say that the deflection is *positive*, then the centre of curvature must lie on the negative side of O A.

$$-\frac{d^2 y}{d x^2} = \frac{P\,y}{E\,I}, \quad \text{multiplying by } \int \frac{d y}{d x}\, d x.$$

$$-\int \frac{d y}{d x} \cdot \frac{d^2 y}{d x^2} \cdot d x = \frac{P}{E\,I} \int y \frac{d y}{d x}\, d x.$$

Therefore, $\left(\dfrac{d y}{d x}\right)^2 = -\dfrac{P}{E\,I}\,(y^2 + c)$.

And, if $\dfrac{d y}{d x} = 0$, then $y = \Delta$, the maximum deflection; hence, $c = -\dfrac{P}{E\,I\,\Delta^2}$.

Therefore, $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{P}{E\,I}\,(\Delta^2 - y^2)$; $dx = \sqrt{\dfrac{E\,I}{P}}\,\dfrac{dy}{\sqrt{\Delta^2 - y^2}}$.

Or, $\displaystyle\int \dfrac{dy}{\sqrt{\Delta^2 - y^2}} = \int \sqrt{\dfrac{P}{E\,I}}\cdot dx.$

Again integrating, $x = \sqrt{\dfrac{E\,I}{P}}\,\sin^{-1}\dfrac{y}{\Delta} + c.$

When $x = 0,\quad y = 0 ;\qquad \therefore\quad c = 0.$

Hence, $y = \Delta \sin\left(x\sqrt{\dfrac{P}{E\,I}}\right),$ $\left\{\begin{array}{l}\text{which is the equation of}\\ \text{the elastic curve.}\end{array}\right.$

When $x = \dfrac{l}{2},\quad y = \Delta.$

Therefore, $\sin\dfrac{l}{2}\sqrt{\dfrac{P}{E\,I}} = 1$; or, $\dfrac{l}{2}\sqrt{\dfrac{P}{E\,I}} = \dfrac{\pi}{2}$; or, $\dfrac{3\pi}{2}$; or, $\dfrac{5\pi}{2}$; &c.

The *least* value of P, and also the *minimum* thrust which will bend the column, is given by the equation—

$$\dfrac{l}{2}\sqrt{\dfrac{P}{E\,I}} = \dfrac{\pi}{2}. \qquad\qquad \text{Or, } P = \dfrac{\pi^2 E\,I}{l^2}. \quad . \quad \text{(VII.)}$$

But $P = p\,A$, and moment of inertia $I = A\,k^2$. Substituting these values in equation (VII.)—

$$p\,A = \dfrac{\pi^2 E\,A\,k^2}{l^2}. \qquad \therefore\quad p = \dfrac{\pi^2 E}{\left(\dfrac{l}{k}\right)^2}.$$

2nd. *When the Column is Fixed at the one end and Hinged at the other.*

Then, $P = \dfrac{\pi^2 E\,I}{\left(\dfrac{2\,l}{3}\right)^2} = \dfrac{9\,\pi^2 E\,I}{4\,l^2}.$ . . . . (VIII.)

3rd. *When the Column is Fixed at Both Ends.*—If the pillar or strut is fixed at both ends, the load which it will stand before yielding is the same as for a strut of half the length hinged at the ends.

Then, $P = \dfrac{\pi^2 E\,I}{\left(\dfrac{l}{2}\right)^2} = \dfrac{4\,\pi^2 E\,I}{l^2}$ . . . . . (IX.)

In this case, the length $= \dfrac{l}{4}$ in (VII.).

It will be noticed, that Euler's equations are for very long struts loaded under ideal conditions, viz. :—

(1) That the column is originally straight and of uniform section ; (2) material of the column is homogeneous ; and (3) the line of action of the load coincides with the centre line or axis of the column.

Professors R. H. Smith and Claxton Fidler have pointed out, that these conditions cannot be realised in practice, as there is nearly always some

eccentricity due to either imperfection in workmanship, want of uniformity in the modulus of elasticity throughout the column, or journal friction occurring in connecting- and eccentric-rods.

If $f_{c\,max.}$ is the compressive or crushing strength of the material of the column and A is the cross-sectional area of the column.

Then, for short columns,

Breaking load $P = f_{c\,max.}$ A.

Also, according to Euler's formula for very long columns—

Breaking load $P = \dfrac{\pi^2 E I}{l^2}.$

Then—

Breaking load $P = f_{c\,max.}\, A = \dfrac{\pi^2 E I}{l^2} = \dfrac{f_{c\,max.}\, A}{1 + f_{c\,max.}\, A \dfrac{l^2}{\pi^2 E I}}.$. (X.)

This equation may be taken as true for columns of all lengths, because if l is small, the denominator is 1 and $P = f_{c\,max.}$ A.

When the length l of the column is great, then we neglect the 1 in the denominator, and—

$$P = \frac{\pi^2 E I}{l^2}.$$

But $I = Ak^2$, where k is the least radius of gyration of the section; then, substituting this value in equation (X.)—

$$P = \frac{f_{c\,max.}\, A}{1 + c\dfrac{l^2}{k^2}}.$$ . . . (XI.)

Where the constant $c = \dfrac{f_{c\,max.}}{\pi^2 E}.$

It is found, that if the constant is calculated from the equation, values are obtained which make the column too strong, because as already noted, that in practice perfect straightness, loading, &c., of the column does not exist.

Therefore, the constants $f_{c\,max.}$ and c are determined from a set of experimental tests, and the formula is treated as an empirical one.

Prof. W. E. Lilly on "The Strength of Solid Cylindrical Round-ended Columns." *—The object in carrying out the experiments was to determine the constants to be used in the formulæ for the design of solid cylindrical round-ended columns. The constants at present in use, more especially in the Rankine-Gordon formula, have been derived from experiments on columns of various figures or shapes of the cross-section, and do not apply with any great accuracy to solid columns. He has

---

* See *Engineering* of November 13th, 1908, for printed copy of paper read at the Dublin meeting of the British Association for the Promotion of Science.

pointed out in some previous papers,* that the wave phenomena which accompany secondary flexure influence the strength of the column, and, therefore, the thickness and the figure or shape of the cross-section must be taken into consideration when estimating its strength.

Tests.—The tests were carried out on samples of cast tool-steel, Bessemer steel, mild-steel shafting, annealed and unannealed, wrought iron, and cast iron. The mode of testing was in all cases the same. Definite lengths of the ½-inch diameter samples were cut off, and the ends rounded in the lathe, the bars being at the same time carefully straightened. They were then placed in the testing machine, the round ends being on steel discs, the surfaces of which were slightly concave, to enable the specimen to be easily adjusted.
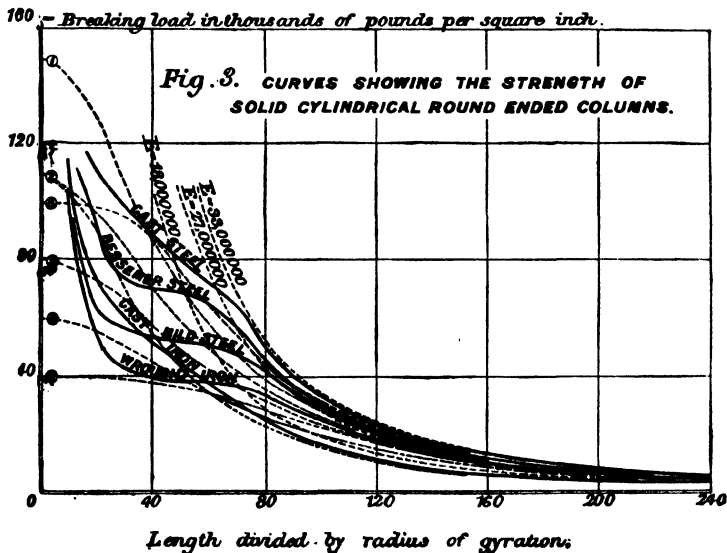
FIG. 3.—CURVES SHOWING THE STRENGTH OF SOLID CYLINDRICAL ROUND-ENDED COLUMNS.

Plotting and Comparison of the Results.—Values of l/ρ, the length divided by the radius of gyration, were plotted on a horizontal line as base, and the values of p, the load in lbs. per square inch on the column, were plotted vertically; the curves were then drawn through the co-ordinate points. The curves obtained are shown in Fig. 3, and the stress-strain diagrams from the tension tests, together with the values for Young's modulus of elasticity being given in Fig. 4. It will be noted that, for

* Students should refer to the following papers by Professor Lilly on "The Strength of Columns," *Proc. Inst. of Mech. Engs.*, June, 1905; "The Economic Design of Columns," *Proc. Inst. C. E. of Ireland*, March, 1907; and "The Design of Struts," *Engineering*, January 10th, 1908.

values of l/ρ greater than 120, there is little difference between the values of the breaking loads obtained on the various materials. For values beyond this limit, the modulus of elasticity governs the strength of all columns, and the values closely approximate to the dotted curves plotted to Euler's formula. For values of l/ρ less than 120 and greater than 40, the ductility and the strength of the material at the elastic limit or near the yield-point play an important part in determining the strength of the columns; since the load producing failure partly stresses the material beyond the elastic limit, and, on taking off the load, the column is found to have received a permanent set or deformation.
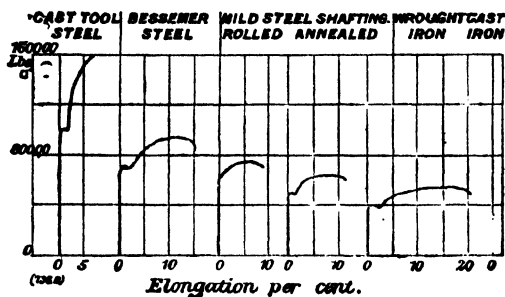
FIG. 4.—PLOTTED RESULTS OF THE ABOVE-MENTIONED MATERIALS AS TESTED.
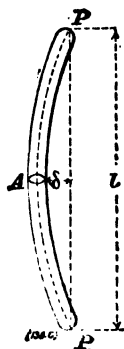
FIG. 5. — SOLID ROUND - ENDED COLUMN UNDER TEST.

	Lbs. per Square Inch.					
Breaking load,	159,000	93,500	74,000	64,000	55,000	33,000
Load at yield-point,	100,000	70,000	64,000	50,000	39,000	..
Young's modulus E,	33,000,000	31,000,000	30,000,000	30,000,000	28,000,000	18,000,000

From a comparison of the curve given for cast iron with the other curves, it will be noticed that a value of $l/\rho = 60$, the curves of wrought iron and cast iron intersect, and, therefore, for values less than, equal to, or greater than this value, a solid column of cast iron is of a strength greater than, equal to, or less than one of wrought iron. Similar remarks also apply for the intersection of the curves for mild steel and cast iron at the value of $l/\rho = 35$. The other curves for the Bessemer steel and cast tool-steel do not intersect the cast-iron curve, and, therefore, columns of these materials are always stronger than those of cast iron. This relation for wrought iron and cast iron was first pointed out by Gordon. His estimate for the critical value of l/ρ was about 80, and somewhat greater than that given above. This is due to the columns having various shapes of the cross-section, the average curve of which intersects at a value of l/ρ greater than that obtained on solid circular columns.

Professor Lilly's Formula.—As shown in a previous part of this Lecture, various formulæ have been proposed at different times, based both upon experimental and theoretical grounds, to determine the strength of columns and struts. Of these he considers the Rankine-Gordon formula to be the best. This formula is of a form proposed by Tredgold on theoretical grounds. It was afterwards revived by Gordon, who determined the approximate values of the constants, and then modified by Rankine, who substituted the value of ρ, the radius of gyration, for the diameter of the cross-section. The formula only applies to solid columns, and requires modification when applied to hollow or irregular forms of cross-section (see pp. 206 to 209).

Professor Lilly, after deriving the Rankine-Gordon formula, modifies it so that the formula becomes—

$$p = \frac{P}{A} = \frac{f}{1 + \frac{f\,l^2}{\pi^2\,E\,\rho^2}\left(\frac{1}{1 + \frac{\pi^2\,E\,\rho}{f\,l^2}}\right)}, \qquad \bullet \quad \bullet \quad \text{(XII.)}$$

where   P = load on the column,
            A = area of cross-section,
            p = load per unit of area,
            f = working strength to compression of the material,
            l = length of column,
            ρ = radius of gyration of cross-section of column,
            E = Young's modulus of elasticity of the material.

He found that, with suitable values of f, the formula gave values which more closely corresponded with his experimental curves. The value of the constants to be used in the derived formula differs from that given for the Rankine-Gordon formula. Young's modulus of elasticity, E, remains the same, but the value of f is about two-thirds of the f of the Rankine-Gordon formula given at top of p. 209.

Advantage of the Derived Formula.—One advantage of the derived formula is, that three points on the curve can be fixed with a fair degree of accuracy from the experiments: thus, when $l/\rho = 0$, the value of $p = f$. Now, if f is taken as the strength of the material near the yield-point, the value of p at the upper limit of the curve is fixed; again, when l/ρ is large, the value of p becomes equal to its Eulerian value, and a second point is thus obtained which fixes the lower limit of the curve. If, now, a column is tested having a value of l/ρ about 80 to 100, the load which will just bend the column can be determined with a fair degree of accuracy, and an experimental value is obtained for a third point on the curve. By giving a suitable value to the constant for f in the second term of the derived formula, the curve can be made to coincide with the experimental value, and a curve is obtained which must approximate closely in its value to the true curve.

Practical Use of Derived Formula for Designing Columns.—Whether the derived formula or the Rankine-Gordon formula be used in designing a column, the areas of the cross-sections obtained are nearly the same in both cases. For this reason, Professor Lilly is of the opinion that little advantage is to be gained in practice by using the derived formula in place of the Rankine-Gordon formula, since the latter errs on the side of safety in giving smaller values of p for values of l/ρ greater than 40; and for columns less than this, which are unusual in practice, it approximates very

closely; and, further, the extreme simplicity of its form must always recommend it to engineers.

Useful Modification of the Rankine-Gordon Formula.—It is convenient for some purposes to put the Rankine-Gordon formula in the following form:—Let $P = f A_1$, where A_1 is the area of cross-section required for a very short column, then equation (X.), p. 212, becomes—

$$\frac{A_2}{A_1} = \frac{A - A_1}{A_1} = \frac{f l^2}{\pi^2 E \rho^2}. \qquad . \qquad . \qquad . \qquad (Xa.)$$

Now, $A_2 = A - A_1$ may be looked upon as the area required to strengthen the column against bending, and A_1 the area required to carry the load. Similarly, for the derived formula, equation (XII.) becomes—

$$\frac{A_2}{A_1} = \frac{A - A_1}{A_1} = \frac{f l^2}{\pi^2 E \rho^2}\left(\frac{1}{1 + \dfrac{\pi^2 E \rho^2}{f l^2}}\right). \quad . \qquad . \qquad . \qquad (XIIa.)$$

From the diagram (Fig. 3) the curves of the two formulæ for annealed mild steel intersect at a value of l/ρ about 60, and, therefore, the areas required for the cross-section of a column would be the same if calculated by either formula. The constants are for $E = 30 \times 10^6$ lbs. per square inch, and for $f_1 = 60,000$ lbs. per square inch for the Rankine-Gordon formula, and 40,000 lbs. per square inch for the derived formula. Substituting these values in equations (Xa.) and (XIIa.), and putting $l/\rho = \sqrt{3,750}$, then, for the Rankine-Gordon formula, $\frac{A_2}{A_1} = \frac{3}{4}$, and, for the derived formula $\frac{A_2}{A_1} = \frac{1}{6}$; that is, of seven parts of the area of the cross-section three parts are required to resist bending and four parts are required to carry the load by the Rankine-Gordon formula, whereas one part is required to resist bending and six parts are required to carry the load by the derived formula. For the larger values of l/ρ the difference rapidly becomes less, but in all cases beyond this value of l/ρ the Rankine-Gordon formula errs on the safe side in giving the larger proportion of the area of the cross-section to resist bending.

Eccentrically-Loaded Columns.[*]—Up to the present time comparatively few tests have been made on the strength of "eccentrically-loaded columns," and the range of these tests has in most cases been too limited to allow of any general deductions being drawn from them. On

---

[*] I am indebted to The Council of The Institution of Civil Engineers, and to the Author, Professor W. E. Lilly, M.A., D.Sc., of the Engineering School in Trinity College, Dublin, for their kind permission to reproduce the following figures and to make certain abstracts from the paper on "Eccentrically-Loaded Columns," see *Proc. Inst. C.E.*, vol. clxxxi., Part III., as issued, Sept., 1910.

These investigations are believed to be the first attempt to carry out a complete series of experiments on eccentrically- and transversely-loaded columns.

Students should endeavour to see and study Prof. Lilly's Articles in *Engineering*, vol. lxxxviii. (1909), p. 1; and vol. lxxxix. (1910), p. 33. Also, his Paper on "The Design of Columns and Struts," see *Engineering*, vol. lxxxvi., p. 670, Nov. 13, 1909.

the other hand, the mathematical treatment of columns under central and eccentric loading has been much discussed, with the result that various formulas have been given for determining their strength. The experimental data required for the verification of the values obtained by the use of these formulas is wanting. For that reason, the Author of this paper carried out some experiments in the Engineering Laboratory of Trinity College, Dublin. He also compares the experimental values obtained with those derived from formulas applicable to the particular cases.

Experimental Tests on Columns.—The experiments were carried out on a 10-ton Wicksteed testing-machine with a number of round bars of nickel steel, Bessemer steel, mild steel, wrought iron, and cast iron of $\frac{1}{4}$ inch diameter; and on mild-steel tubes of $\frac{3}{4}$ inch diameter, of 18 and 23 S.W.G. thick. These bars and tubes were tested as columns under (i.) central, (ii.) eccentric, and (iii.) transverse and axial loading, all as shown by the next Fig. 6.

Manner of Preparing the Columns, and of Conducting the Tests.— Definite lengths of bars and of tubes were cut off and carefully straightened. The grips were then applied as required to these columns, and they were placed in the testing machine with the knife-edges resting in the grooves at the required eccentricity. The test was then carried out in the usual manner, and the load producing failure was noted. The eccentric loads were in all cases applied at distances of $\frac{1}{8}$ inch and $\frac{1}{4}$ inch from the centre of the column. Then the values of em/ρ^2 are 2 and 4 respectively.

For the centrally-loaded columns the same procedure was adopted. For some of the tests the grips and knife-edges were used. But for others, round-ended columns with ends resting on steel discs were adopted.

For the combined transverse

FIG. 6.—METHOD OF APPLYING LOAD BY GRIPS; ALSO TRANSVERSE LOADING.

and axial loadings the grips and knife-edges were used. Then the load was applied at zero eccentricity to the column. The transverse concentrated load was first applied to the column by hanging weights on a cord passing over a fixed pulley. This cord was connected to the column at right angles to the middle of its length. The vertical load was then applied, and the test carried out in the usual way. The apparatus is illustrated in the accompanying Fig. 6.

Manner of Recording the Results obtained from the Experiments on the several Diagrams.—Values of l/ρ (or the length divided by the radius of gyration) were plotted on a horizontal line as base, against the value of p

vertically (for the stress in pounds per square inches on the column). Distinctive symbols were used for the values obtained from the respective tests on the central, eccentric, and combined transverse and axial loadings. On the diagrams the particulars are also given of the material, the eccentricities of loading of the columns, and the constants for the formulæ used in plotting the curves. The diagrams are thus to a large extent self-explanatory.　·ᵎ

Let p denote the load per square inch on the column in lbs.

,,	P	,,	,,	total load on the column in lbs.
,,	A	,,	,,	area of the cross-section in square inches.
,,	f	,,	,,	compressive stress of the material in lbs. per square inch.
,,	l	,,	,,	length of the column in inches.
,,	m	,,	,,	distance of outer fibre from neutral axis—*i.e.*, the outer radius of the circular cross-section, in inches.
,,	ρ	,,	,,	radius of gyration of the cross-section about a diameter in inches.
,,	t	,,	,,	thickness of cross-section in inches.
,,	E	,,	,,	Young's Modulus of Elasticity in lbs. per square inch.
,,	p_0	,,	,,	outer fibre stress due to bending by transverse load on the column in lbs. per square inch.
,,	k	,,	a constant $= \frac{1}{4}$ for mild steel.
,,	e	,,	the eccentricity of loading—*i.e.*, the distance of the load from the centre of the ends of the column in inches.
,,	c	,,	a constant $= \dfrac{f}{\pi^2 E}$.

Various Formulæ—*Euler's Formula.*—The investigation underlying the derivation of formulæ for the strength of columns under axial or eccentric loads, starts from the analysis originally given by Euler, and is given in this Lecture.

$$P = \frac{\pi^2 E I}{l^2} \quad \text{(See Eq. VII.,).}$$

Rankine-Gordon Formula.—If we consider one part of the area of the cross-section to transmit the direct thrust and the other part to transmit the bending moment. Let $P = f A_1$, and $A_1 + A_2 = A$, where A_1 is that part of the cross-sectional area transmitting the thrust, and A_2 the other part transmitting the bending moment; the value of I will be $A_2 \rho^2$. Substituting in Euler's formula these values give—

$$\frac{f l^2}{\pi^2 E \rho^2} = \frac{A_2}{A_1} \text{ or } p = \frac{P}{A} = \frac{f}{1 + \dfrac{f l^2}{\pi^2 E \rho^2}} \quad \begin{array}{l}\text{(See Eq. X}a\text{., Lect. VIII.,}\\ \text{Vol. II.)}\end{array}$$

which is the form of the Rankine-Gordon formula for columns (see pp. 208 and 212). Reverting to Fig. 1, in this Lecture, consider the equilibrium of the central cross-section, then from the theory of bending, the following relation is obtained :—

$$\frac{M}{I} = \frac{p_1}{m} \quad \text{(See Eq. V., Lect. III., Vol. II.),}$$

where p_1 is the stress due to bending on that fibre at the greatest distance m from the neutral axis. At the centre $M = P \Delta$, where Δ is the deflection at the centre; and putting A equal to the total area of the cross-section, then $I = A \rho^2$ and $P = p A$, and on substitution the relation obtained is $\dfrac{p_1}{p} = \dfrac{\Delta m}{\rho^2}$. If, as before, the cross-section is considered as being made up of two areas, one of which A_1 transmits the direct stress and the other A_2 the bending stress, then if the outer fibre stress due to bending is equal to the direct stress, $P = p A$ may be put equal to $p_1 A_1$ and $I = A_2 \rho^2$,

or
$$\frac{A_2}{A_1} = \frac{\Delta m}{\rho^2} = \frac{p_1}{p}. \qquad . \qquad . \qquad . \qquad . \qquad \text{(XIII.)}$$

This relation obtains because any alteration of the breadth of a given cross-section does not alter the value of ρ.

Adding unity to both sides of this equation, and putting $p_1 + p = f$. then

$$p = \frac{f}{1 + \dfrac{\Delta m}{\rho^2}}. \qquad . \qquad . \qquad . \qquad . \qquad \text{(XIV.)}$$

The difficulty is, that the solution of equation (XIII.) depends upon the validity of assuming such distribution of stress on the cross-section, and upon determining what function of the length applies to Δ; but so far the latter can only be approximated to. For the Rankine-Gordon formula $\Delta = \dfrac{f l'}{\pi^2 E m}$, and the assumption is legitimate when comparing columns of the kind already mentioned, as it assumes that the deflection of the column under the breaking load stresses the material up to its compressive stress f. For long columns Euler's formula leads to the same result.

The Rankine-Gordon formula, as thus derived, gives the first approximation and errs on the side of safety, as will be evident on comparing the experimental results with the curves in the figures; further, it leads to a rational method of evaluating the shear stress in columns, from which the proportions of the bracing or web in built-up columns can be determined; and lastly, the simplicity of its form must always recommend it to engineers.

Prof. R. H. Smith's Formula * *for Eccentrically-Loaded Solid Columns.*— Let the load P be applied to the column with an eccentricity e from the centre of the column, then Euler's formula—*Eq.* (VII.)—becomes

$$\frac{P(y + e)}{E I} = \frac{1}{R} = -\frac{d^2 y}{d x^2} \qquad . \qquad . \qquad . \qquad \text{(XV.)}$$

or
$$\frac{d^2 y}{d x^2} + n^2 y + n^2 e = 0, \text{ where } n^2 = \frac{P}{E I}.$$

---

* See *The Engineer*, vol. lxiv. (1887), p. 304, for article on " Struts."

The solution of this differential equation is—

$$y = - e + A \cos n x + B \sin n x.$$

Since y has the same value for $+ x$ or $- x$,

$$y = - e + A \cos n x,$$

And when $\qquad x = 0, y = \Delta - e,$ giving $A = \Delta:$

Hence $\qquad\qquad\qquad y = - e + \Delta \cos n x.$

Again, when $\qquad\qquad x = \dfrac{l}{2},$ then $y = 0,$

giving $\qquad\qquad\qquad e = \Delta \cos \dfrac{n l}{2} . \quad . \quad . \quad . \quad$ (XVI.)

For the case when the eccentricity of loading is zero,

$$0 = \Delta \cos \frac{n l}{2} = \Delta \cos \sqrt{\frac{P}{E I}} \cdot \frac{l}{2}, \text{ giving Euler's formula.}$$

From eq. (XVI.)

$$\Delta = \frac{e}{\cos \dfrac{n l}{2}} = \frac{e}{\cos \sqrt{\dfrac{P}{E I}} \cdot \dfrac{l}{2}},$$

and by substitution in eq. (XIV.) for Δ.

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2} \cdot \dfrac{1}{\cos \sqrt{\dfrac{P}{E I}} \cdot \dfrac{l}{2}}} . \quad . \quad . \quad \text{(XVII.)}$$

Modified Form of the Rankine-Gordon Formula for Eccentrically-Loaded Solid Columns.—Referring back to the interpretation of Euler's formula given in eq. (Xa.), the above eq. (XVII.) can be written

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2} \cdot \dfrac{1}{\cos \sqrt{\dfrac{A_1 f}{A_2 E \rho^2}} \cdot \dfrac{l}{2}}} .$$

If now in eq. (XVII.) the term containing the cosine is expanded, then

$$\cos \sqrt{\frac{P}{E I}} \cdot \frac{l}{2} = 1 - \frac{P l^2}{4 \lfloor 2 E I}, \&c.;$$ and taking these first two terms as an approximation, eq. (XVI.) becomes

$$\Delta = \frac{e}{1 - \dfrac{P l^2}{8 E I}} .$$

If the approximation is made of putting π^2 for 8, and the substitution is made as in eq. (Xa.), then

$$\frac{f\,A_1 l^2}{\pi^2 E\,A_2 \rho^2} \text{ may be put for } \frac{P\,l^2}{8\,E\,I}, \text{ giving } \Delta = \frac{e}{1 - \dfrac{f\,A_1 l^2}{\pi^2 E\,A_2 \rho^2}},$$

and by substitution in eq. (XIII.)

$$\frac{A_2}{A_1} = \frac{p_1}{p} = \frac{\Delta\,m}{\rho^2} = \frac{\dfrac{e\,m}{\rho^2}}{1 - \dfrac{f\,A_1 l^2}{\pi^2 E\,A_2 \rho^2}},$$

when the modified form of the Rankine-Gordon formula is obtained :—

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2} + \dfrac{f\,l^2}{\pi^2 E \rho^2}}. \qquad . \qquad . \qquad . \qquad . \quad \text{(XVIII.)}$$

Prof. Perry's,[*] *Neville's,*[†] *Moncrieff's*[‡] *and Prof. Fidler's*[§] *Formulæ.*—
Prof. Perry's approximation assumes that $\cos\sqrt{\dfrac{P}{E\,I}} \cdot \dfrac{l}{2} = \dfrac{1\cdot2}{1 - \dfrac{P\,l^2}{\pi^2 E\,I}}$, giving
on substitution in eq. (XIII.), as before

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2} \cdot \dfrac{1\cdot2}{1 - \dfrac{P\,l^2}{\pi^2 E\,I}}} = \frac{f}{1 + 1\cdot2\,\dfrac{e\,m}{\rho^2} + \dfrac{p_1 l^2}{\pi^2 E\,\rho^2}}. \qquad . \quad \text{(XIX.)}$$

Mr. E. A. Neville's approximation assumes a form giving on substitution in eq. (XIII.)

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2} \cdot \dfrac{1}{1 - \dfrac{P\,l}{9\,E\,\rho^2}}} = \frac{f}{1 + \dfrac{e\,m}{\rho^2} + \dfrac{p_1 l^2}{9\,E\,\rho^2}}. \qquad . \qquad . \quad \text{(XX.)}$$

Mr. J. M. Moncrieff's formula is derived by considering the curve of an eccentrically-loaded column to be a *parabola*. The method of deriving it leads to its form being similar to the formulæ already given, and on substitution in eq. (XIII.) it gives

$$p = \frac{f}{1 + \dfrac{e\,m}{\rho^2}\left\{\dfrac{1 + \dfrac{0\cdot2\,p\,l^2}{9\cdot6\,E\,\rho^2}}{1 - \dfrac{p\,l^2}{9\cdot6\,E\,\rho^2}}\right\}} = \frac{f}{1 + \dfrac{e\,m}{\rho^2}\left(1 + \dfrac{0\cdot2\,p\,l^2}{9\cdot6\,E\,\rho^2}\right) + \dfrac{p_1 l^2}{9\cdot6\,E\,\rho^2}}. \quad \text{(XXI.)}$$

---

[*] See *The Engineer*, vol. lxii. (1886), p. 464, for article on " Struts " by W. E. Ayrton and J. Perry.

[†] See Technical Paper, No. 129, Government of India, Simla, 1902, for " Note on Euler's Formula," by E. A. Neville.

[‡] See *Trans. Am. Soc. C.E.*, vol. xlv., p. 334, and *Engineering*, vol. lxxiii. (1902), p. 731.

[§] See *Proc. Inst. C.E.*, vol. lxxxvi., p. 261, paper on " The Practical Strength of Columns and Braced Struts," by Prof. Claxton Fidler.

Professor Claxton Fidler has given a formula rather similar to the preceding, which takes into consideration a variation of the modulus of elasticity of the material of the column. Mr. Neville has shown that this assumption is equivalent to considering the column to have an initial deflection, with the result that the values obtained by its use do not differ greatly from those obtained from the formulæ already given.

Prof. Lilly's Formula for Transversely-Loaded Solid Columns.—The Rankine-Gordon formula, and the modification of it for eccentrically - loaded columns given in eq. (Xa.), may be simply derived by considering the curve of deflection to be a parabola, as follows :—

From eq. IV., Lect. III., Vol. II., and eq. (XIII.)

$$\frac{f\,l^2}{\pi^2 \mathrm{E}\,m} \cdot \frac{m}{\rho^2} = \frac{\mathrm{A}_2}{\mathrm{A}_1} = \frac{\Delta\,m}{\rho^2},$$

and this relation will be true for any length of column with its corresponding deflection. Hence, from Fig. 7,

$$(\Delta - e)\frac{m}{\rho^2} = \frac{f\,l^2}{\pi^2 \mathrm{E}\,m} \cdot \frac{m}{\rho^2},$$

and substituting for Δ in eq. (XIII.) gives eq. (XVIII.) again.

For the case of a *transversely-loaded* column the above formula is transformed, thus—

$$\frac{\Delta\,m}{\rho^2} = \frac{e\,m}{\rho^2} + \frac{f\,l^2}{\pi^2 \mathrm{E}\,\rho^2},$$

but $\dfrac{\Delta\,m}{\rho^2} = \dfrac{\mathrm{A}_2}{\mathrm{A}_1}$, and multiply both sides by $\dfrac{\mathrm{A}_1}{\mathrm{A}_2}$,

$$\frac{\mathrm{A}_1}{\mathrm{A}_2} \times \frac{\mathrm{A}_2}{\mathrm{A}_1} = 1 = \frac{\mathrm{A}_1}{\mathrm{A}_2}\left(\frac{e\,m}{\rho^2} + \frac{f\,l^2}{\pi^2 \mathrm{E}\,\rho^2}\right).$$

FIG. 7.—SKETCH OF ECCENTRICALLY-LOADED COLUMN.

Also, $\dfrac{\mathrm{P}\,e}{\mathrm{I}} = \dfrac{f\,\mathrm{A}_1 e}{\mathrm{A}_2 \rho^2} = \dfrac{p_0}{m}$;

hence $\dfrac{p_0}{f} + \dfrac{f\,l^2}{\pi^2 \mathrm{E}\,\rho^2} \cdot \dfrac{\mathrm{A}_1}{\mathrm{A}_2} = 1$; or $\left(\dfrac{f}{f - p_0}\right)\dfrac{f\,l^2}{\pi^2 \mathrm{E}\,\rho^2} = \dfrac{\mathrm{A}_2}{\mathrm{A}_1}$,

giving from eq. (XIII.),

$$p = \frac{f}{1 + \dfrac{f\,l^2}{\pi^2 \mathrm{E}\,\rho^2} \cdot \dfrac{f}{f - p_0}}. \qquad \bullet \qquad \bullet \qquad \text{(XXII.)}$$

The curves numbered 5 on Figs. 9 and 10 have been plotted to it, the value of p_0 being taken equal to $f/2$ in all cases.
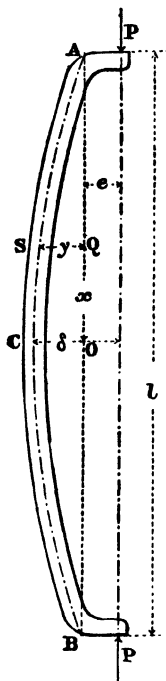
To determine the concentrated load required to produce a given fibre-stress p_0, a supplementary calculation is required. Let

$$\frac{f \, \mathrm{L}^2}{\pi^2 \mathrm{E} \, \rho^2} = \frac{f \, l^2}{\pi^2 \mathrm{E} \, \rho^2} \cdot \frac{f}{f - p_0}.$$

And substituting the value of $p_0 = f/2$, then $\mathrm{L} = \sqrt{2} \, l$. Now the length L may be looked upon as representing a column which would carry the same column load as the transversely-loaded one; hence from eq. IV., Lect. III., Vol. II.,

$$\frac{\mathrm{A}}{\mathrm{A}} = \frac{f \, \mathrm{L}^2}{\pi^2 \mathrm{E} \, \rho^2} = \frac{2 f \, l^2}{\pi^2 \mathrm{E} \, \rho^2}, \text{ or } \mathrm{A}_2 = \frac{\mathrm{A}}{1 + \dfrac{\pi^2 \mathrm{E} \, \rho^2}{2 f \, l^2}} \qquad \textbf{(XXIII.)}$$

From eq. (V.), Lect. III. Vol., II., $\dfrac{p_0}{m} = \dfrac{\mathrm{M}}{\mathrm{I}}$, also $\mathrm{I} = \mathrm{A}_2 \rho^2$, and substituting the value of A_2 from eq. (XXIII.),

$$\mathrm{M} = \frac{p_0 \rho^2}{m} \cdot \frac{\mathrm{A}}{1 + \dfrac{\pi^2 \mathrm{E} \, \rho^2}{2 f \, l^2}} \qquad \bullet \qquad \bullet \qquad \bullet \qquad \textbf{(XXIV.)}$$

For the *concentrated load*, $\mathrm{M} = \dfrac{\mathrm{W} \, l}{4}$, giving

$$\mathrm{W} = \frac{p_0 \rho^2}{m} \cdot \frac{\mathrm{A}}{\dfrac{l}{4} + \dfrac{\pi^2 \mathrm{E} \, \rho^2}{8 f \, l}} \qquad \bullet \qquad \bullet \qquad \bullet \qquad \textbf{(XXV.)}$$

From eq. (XXIV.), if p_0 is given, M for a given distribution of loading can be determined, and conversely.

Professor Lilly's Formulæ for Hollow Columns.[*]—The formulæ given in this article for hollow columns are based upon some experimental and analytical work carried out by Prof. Lilly, in which it was shown that a short, hollow column when tested under compression fails by secondary flexure or wrinkling, the column breaking up into a series of waves. From a comparison of the results obtained from his tests it was shown that the load producing failure becomes smaller, as the diameter of the column increases and the thickness decreases, and it is not until the length of the column becomes less than the length of one wave that the load producing failure approaches the compressive strength of the material. Hence the true strength to compression of the column is the load which produces the wave formation. From the experiments, and also from the analysis of these waves, it was found that the wave length varied as the square root of the area of the cross-section. This result leads to the following equation for the limiting load :—

$$f_1 = \frac{f}{1 + k \dfrac{\ell}{t}}, \qquad \bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \textbf{(XXVI.)}$$

---

[*] *Proc. Inst. of Mech. Engs.* (1905), p. 697 ; also, *Trans. of Inst. C.E. of Ireland,* vol. xxxiii. (1907), pp. 67 and 181 ; vol. xxxiv., p. 27.

The value of the coefficient k depends upon the shape or figure of the cross-section, and the ratio ρ/t depends upon its diameter and thickness. For mild-steel hollow columns of circular cross-section $k = \frac{1}{4}$. Further particulars of these constants and the experiments on which they are based are given in the original Papers referred to in the footnote.

On substitution of the above result in the Rankine-Gordon formula the following modified formulæ are obtained :—

For Centrally-loaded hollow columns—

$$p = \frac{P}{A} = \frac{f}{1 + k\frac{\rho}{t} + \frac{fl^2}{\pi^2 E \rho^2}} \qquad \bullet \quad \bullet \quad \bullet \quad \text{(XXVII.)}$$

For Eccentrically-loaded hollow columns—

$$p = \frac{P}{A} = \frac{f}{\left(1 + k\frac{\rho}{t}\right)\left(1 + \frac{em}{\rho^2}\right) + \frac{fl^2}{\pi^2 E \rho^2}} \qquad \bullet \quad \bullet \quad \text{(XXVIII.)}$$

Comparison of the Results obtained by the several Formulæ.—The solid circular section has been taken in each case, and the eccentricity e is assumed at $\frac{1}{20}$ and $\frac{1}{100}$ of the radius respectively.

Two curves are plotted from Prof. Smith's formula in Fig. 8, and it will be noticed that as the value of e becomes very small the curve approximates to a straight line for values of l/ρ less than 65, and to Euler's curve for values of l/ρ greater than 65. This is equivalent to assuming for columns of medium length that the deflection is proportional to some less power than the square of the length. From the views already expressed, the values obtained by its use are not theoretically more correct than those given by the Rankine-Gordon formula.

The values obtained by calculation from Professor Perry's, Mr. Neville's, and Mr. Moncrieff's formulæ are shown plotted in the figure, and they can be compared with the curves obtained from Prof. Smith's formula, to which they approximate very closely.

It should be noted that the formulæ given in equations (XIX.), (XX.), and (XXI.) cannot be solved unless some assumption is made as to the value either of the fibre bending stress p_1, or of the eccentricity. If p_1 is put equal to f the formulæ differ but little from the modified form of the Rankine-Gordon formula.

Various other formulæ, known as " *straight-line* " *formulæ*, for columns, are at present in use, but they are not based upon the usually accepted theory of flexure. Further, the values obtained from their use do not agree with the values obtained from experiments when carried out over a large range of the length.

The comparison of the values derived from the formulæ given with the results obtained from the experimental investigation are in general in close agreement, the experimental values being greater than the calculated values. For this reason Professor Lilly considers, that the following general formula may be used with confidence in solving most of the problems which occur in the design of columns, since the formulæ given are only particular cases :—

$$p = \frac{P}{A} = \frac{f}{\left(1 + k\frac{\rho}{t}\right)\left(1 + \frac{em}{\rho^2} \cdot \frac{f}{f - p_0}\right) + \frac{fl^2}{\pi^2 E \rho^2} \cdot \frac{f}{f - p_0}}. \quad \text{(XXIX.)}$$

Fig. 9.

NICKEL-STEEL SOLID CYLINDRICAL COLUMNS

VALUES OF l/p

Fig. 10.

MILD-STEEL SOLID CYLINDRICAL COLUMNS

VALUES OF l/p

Fig: 11.

MILD-STEEL TUBULAR COLUMNS

⅞" Diameter, 0.048" thick – 18 S.W.G
• = Central Loads
◆ = Eccentric Loads
The curves for the formulæ are
f = 20,000 E = 30,000 $\frac{d}{d+e}$ = 260

VALUES OF l/ρ

Fig: 8.

THE STRENGTH OF COLUMNS

Values derived from the Formulæ of
Euler, Rankine-Gordon, and Smith shown by curves
Perry: – Rankine –● Moncrieff –○ shown by plotted values
In all cases f = 67,600 E = 30,000,000
The eccentricity e = l/ρ and l/ρ of the radius.

VALUES OF l/ρ

The formula is of simple form, easily derived, and leads to consistent results, whereas Euler's formula as usually interpreted does not.

Tables of Constants for the Formulæ and Values obtained from the Tension Tests on the Materials.—The constants for the formulæ to which the several curves have been plotted are given in the following table :—

Material.	Compressive Strength (f).	Modulus of Elasticity.	Round-ended Columns (c).
	Lbs. per sq. in.	Lbs. per sq. in.	
Nickel steel, . .	120,000	31,000,000	$\frac{1}{3100}$
Bessemer steel, . .	110,000	31,000,000	$\frac{1}{3100}$
Mild steel, . .	80,000	30,000,000	$\frac{1}{3000}$
,, annealed, .	60,000	30,000,000	$\frac{1}{3000}$
Wrought iron, annealed,	55,000	26,000,000	$\frac{1}{3100}$
Cast iron, . . .	110,000	14,000,000	$\frac{1}{1400}$

The values obtained from the tension tests on the various materials are as follows :—

Material.	Tensile Breaking Stress.	Stress at Yield-point.	Elongation in 10 Inches.
	Lbs. per sq. in.	Lbs. per sq. in.	Per cent.
Nickel steel, . .	109,000	80,000	12
Bessemer steel, . .	93,500	70,000	14
Mild steel, . .	74,000	64,000	8
,, annealed, .	64,000	50,000	12
Wrought iron, annealed,	55,000	39,000	20
Cast iron, . . .	33,000	..	..

Notes on the Several Diagrams.—In Fig. 9 are shown the values obtained from the tests carried out on a sample of *nickel steel* containing 3 per cent. nickel. These tests appear to be the first series of column-tests carried out on this material, and the constants as determined will be of interest now that it is being used in structural engineering.

The values for *mild steel* are shown in Fig. 10. These are of interest, since two complete sets of experiments are shown, one set being carried out on the bars in the condition " as received," and the other set after the bars had been " annealed." The sample used was that known as bright, rolled, mild-steel shafting, and owing to the previous mechanical treatment it had received, the material was in state of stress, with the result that when it was tested the values obtained were higher than those obtained from it in the annealed condition. These two sets of experiments show the marked influence of permanent stress upon the strength of ductile materials when tested as columns.

In Fig. 11 is shown the values obtained from tests carried out on a mild-steel tube of ¾ inch diameter, and 18 S.W.G. thick. These have been selected from a number of tests that were carried out on tubes of different diameters and thicknesses to show the influence of secondary flexure on

2

the strength of such columns. In the case of the eccentric loading there is little variation in the loads producing failure for a large range of the length, which is due to the small deflection of the column under the applied load compared with the eccentricity of loading.   The value of p/t is 5 for the 18 S.W.G. tube, and experiments with an eccentricity of $\frac{1}{8}$ inch and of $\frac{1}{4}$ inch were made.

In addition to the above, tests were carried out on *annealed wrought-iron columns.* The iron was annealed in order to remove any permanent stress and thus obtain more consistent results. In addition to the other tests, one set of bars was tested as round-ended columns bent with an initial deflection of $\frac{1}{4}$ inch. These latter tests were carried out to compare the strength of a bent column with a straight column eccentrically loaded under the same conditions. The results obtained were in close agreement.

EXAMPLE II.—A rod of steel, 4 inches diameter, 10 feet long, acts as a strut, but the resultant load does not act exactly at the centre of each end. The inexactness of loading is, say, h ; imagine it the same at both ends. If the greatest stress in the material is not to exceed 20,000 lbs. per square inch, and if Young's modulus is 3×10^7 lbs. per square inch, find the greatest loads for the strut in three cases, first when $h = 0.1$ inch, second when $h = 0.01$ inch, third when h is 0.
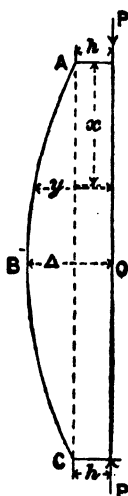
FIG. 12.—SKETCH OF STEEL ROD LOADED ECCENTRICALLY.

ANSWER.—Let l = Length of the unbent strut.
　　　,, P = Load applied.
　　　,, h = Distance of P from neutral axis at either end of the strut.

Consequently, if A B C represents the neutral axis of the strut, and the load line P P be chosen as the axis of x, the mid-point O of P P is the origin and O B the axis of y. The differential equation to the curve into which the strut is bent will be—

$$E\,I\,\frac{d^2 y}{d\,x^2} = -\,P\,y. \qquad . \qquad . \quad (1)$$

Where E is Young's modulus of elasticity of the material, and I is the moment of inertia of the cross-section of the strut.

Equation (1) is derived from the consideration that the curvature of the neutral axis at any point is proportional to the bending moment at that point. This latter is proportional to y, which is the distance of the point from the line of action P P of the applied forces.

It is also known that the curvature at any point on the curve

$$y = \Delta \cos \frac{\pi x}{L} \qquad . \qquad . \quad (2)$$

is proportional to y if Δ is small ; so that we may assume (2) to be the equation to the neutral axis of the strut

The cosine curve is shown in preference to the sine curve, since y must obviously have the same value for numerically equal positive and negative values of x. L is a constant to be determined, and we know that it cannot be equal to l unless the strut is loaded at the centre, for only in this latter case will the neutral axis cut the line P P at $x = \pm \frac{l}{2}$.

Substituting from equation (2) in (1), we obtain, after simplifying—

$$P = \frac{\pi^2 E I}{L^2}. \qquad . \qquad . \qquad . \qquad . \qquad (3)$$

When the strut is centrally loaded, and $c = l$, then equation (3) gives Euler's formula for the "crippling load"—*i.e.*, the least load which will cause the strut to bend. It is easily seen that when P is less than the value given by equation (3), and c has a constant value $= l$, equation (1) can only be satisfied by (2) when $\Delta = 0$.

In this case of the eccentrically loaded strut, by substituting $x = l/2$ and $y = h$ in (2), we get—

$$y = \Delta \cos \frac{\pi x}{L}; \ i.e., \ h = \Delta \cos \frac{\pi l}{2 L}. \qquad . \qquad . \qquad (4)$$

Let $f_b =$ The maximum stress produced in the fibres of the middle section of the strut due to bending.

,, $f_{e \, max.} =$ The maximum compressive stress.

,, $f =$ Maximum compressive stress which the fibres of the strut can withstand without passing the elastic limit.

,, $r =$ The radius of the cross-section of the strut.

,, $A =$ The cross-sectional area of the strut $= \pi r^2$.

Then, $P \Delta = \frac{f_b I}{r}; \ \text{or}, f_b = \frac{P \Delta r}{I}.$

Also, the uniform compressive stress due to the load P is $f_{e \, max.} = \frac{P}{A}.$

Hence the greatest compressive stress on the strut is—

$$f = f_{e \, max.} + f_b = \frac{P}{A} + \frac{P \Delta r}{I} = P \left(\frac{1}{A} + \frac{\Delta r}{I} \right).$$

$$\therefore \quad P = \frac{f A I}{I + A \Delta r}. \qquad . \quad (5)$$

Substituting this value for P in equation (3), and simplifying, we obtain—

$$\frac{E \pi^2}{f A} = \frac{L^2}{I + A \Delta r}. \qquad . \quad (6)$$

Equations (6) and (4) form two simultaneous equations in the unknown quantities Δ and l. When the deflection Δ has been calculated from these equations, the maximum permissible load can be obtained by substituting in (5).

The simultaneous equations (6) and (4) can be solved graphically. The corresponding curves are plotted, and the co-ordinates of their point of intersection give the required values of Δ and L. The curve A B in the figure is the graph of the equation—
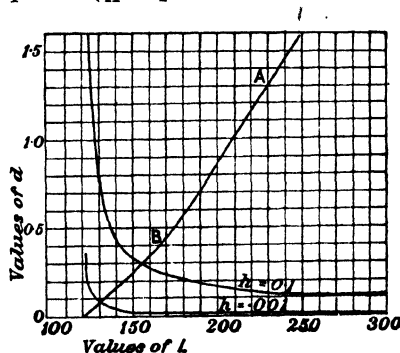
FIG. 13.—PLOTTED RESULTS, SHOWING THE RELATIONS OF THE DEFLECTIONS TO THE LENGTHS OF THE STRUT.

$$1,177 = \frac{L^2}{12 \cdot 56 + 25 \cdot 12 \, \Delta}.$$

In plotting this curve, various values of L greater than $l = 120$ inches were assumed, and the corresponding values of deflection Δ were calculated. The other two curves in the figure are graphs of the equation—

$$\Delta = \frac{h}{\cos \dfrac{\pi l}{2 L}},$$

where $l = 120$, and h has the values ·1 and ·01 respectively.

It is found, that the curve for $h = ·1$ cuts A B at a point corresponding to $\Delta = ·295$. Substituting this value in (5), we get $P = 1·58 \times 10^5$ lbs. The curve for $h = ·01$ intersects the curve A B at a point corresponding to $\Delta = ·085$. Substituting in (5), we obtain $P = 2·14 \times 10^5$.

When the loading is central, $h = 0$, and we must substitute $L = l = 120$ in (3). This gives $P = 258,000$ lbs. for the "crippling load," by Euler's formula—i.e., the least load which will produce bending.

The maximum load which can be sustained by the strut without crushing is—

$$f A = 2 \times 10^4 \times 12·56 = 251,200 \text{ lbs.}$$

Therefore, when the strut is loaded centrally, it breaks from crushing, without bending, when the load is equal to 251,200 lbs.

Thus—   When $h = 0$, $P = 251,200$ lbs.
            „ $h = 0·01$, $P = 214,000$ lbs.
            „ $h = 0·1$, $P = 158,000$ lbs.

In order that Euler's equation for the crippling load may not be misunderstood, a short explanation may be useful. If the central load is less than that calculated from (3), when $L = l$, the strut will not bend, though it may fail from crushing.

What happens when the load is equal to that given by (3)?

The left-hand side of equation (1) represents the resisting moment at a section $x y$ of the strut, while the right-hand side represents the turning moment at the same section. The equation to the neutral axis is given by equation (2), and, substituting it in (1), we obtain—

$$\frac{\pi^2 E I}{L^2} \Delta \cos \frac{\pi x}{L} = P \Delta \cos \frac{\pi x}{L}. \qquad . \qquad . \qquad . \quad (7)$$

When the loading is central, L has the constant value l; and, since both sides of equation (7) are proportional to Δ, it follows that if bending increases, the moment producing this bending increases at the same rate as the moment to stresses in the fibres of the strut which opposes bending. Consequently, the load given by (3), which can just cause bending to commence, is sufficient to bend the rod to any extent. When the fibres are stressed beyond their elastic limit, the resisting moment increases more slowly than the bending moment, because the deflection increases and the strut breaks. Hence we see, that when the load attains the value given by (3) the strut becomes *unstable*, whatever may be its strength under crushing or tension. When the strut is eccentrically loaded, equation (4) shows, that L must diminish as Δ increases, so that the left-hand side of (7) increases at a greater rate than the right-hand side, and *stability* is secured unless the stresses in the fibres exceed the limits of elasticity.

LECTURE VIII.—QUESTIONS.

1. A cast-iron column is straight before being loaded, will it remain so when loaded? If so, why? If not, why not?

2. Illustrate and prove Rankine's formulæ for the three cases of fixing vertical columns.

3. Illustrate and prove Gordon's formulæ for the three cases of fixing vertical columns. State the difference between Rankine's and Gordon's formulæ.

4. Why do long columns fail to support loads? Prove Euler's formulæ for long columns.

5. What are the ideal conditions of loading which satisfy Euler's equations for the loading of long columns, and why?

6. A cast-steel column, 6 inches outside, 4 inches inside diameter, and 20 feet in length, is used as a strut, with the direction of the resultant load (1) centrally and (2) ·5 inch from, but parallel to, its central axis. Let the greatest stress per square inch be 25,000 lbs. and Young's modulus 3×10^7 lbs. per square inch; find the greatest load which the column will sustain in each case. Plot your results to scale.

7. A structure has a hollow circular section 10 inches outside diameter and 8 inches inside. The resultant of all the loads and supporting forces acting on one side of the section has a component of 30 tons normal to the section, and it acts at 2 inches from the centre; find the maximum and minimum stresses in the section.

8. A solid wrought-iron cylindrical strut is 8 feet 4 inches long, and has its ends solidly built in : assuming it has only a pure compressive load of 24 tons to support, what must be the diameter if the working load is only ⅛th of the collapsing load?

9. It is known that a solid cast-iron column of 4 inches diameter and 7 feet in length would collapse under a load of 17·5 tons per square inch ; what total load could you allow the column to support with a factor of safety of 8?

10. How much would a cast-iron column 12 feet long shorten under a load of 3¼ tons per square inch, if it is a hollow section 8 inches in external diameter and 1 inch thick? The modulus of elasticity of cast iron in compression is 12,500,000 lbs. per square inch. What total compressive load will this column support under the above stress per square inch?

11. Explain why the resistance of a long strut depends more on the stiffness of the material than on its strength. Quote any formula which is used in the design of long struts.

1. A vertical mild steel tube of 6 inches external diameter, ½ inch thick, is securely bedded in the ground. Its height above ground is 10 feet, and it is subjected at the upper end to a horizontal pull of 1,500 lbs. Calculate the maximum stress at the ground section and the deflection at the top. Take E as 30,000,000 lbs. per square inch. *Ans.* 7·3 tons per square inch; deflection = ·875 inch.

2. Criticise briefly Euler's, Rankine's, and Gordon's formulæ for struts. Which do you prefer to use, and for what reasons?

3. A number of connected loads roll across a bridge. Explain carefully how you would find the maximum bending moment at any section.

4. A masonry column is circular in section. Show that, if the line of pressure cuts the section at a point which is distant more than ⅛ of the diameter from the centre, there will be tensile stress over a portion of the section.

5. A rolled **T** bar, 4 inches × 4 inches × ½ inch, is to be used as a strut. Find the moment of inertia I, and the radius of gyration k of the section about a neutral axis parallel to the top-table, and about another at right angles to the first. *Ans.* I = 5·56 and 2·7; k = 1·22 and ·85.

6. A slender rod of steel, 5 feet long, and either square or cylindrical in section is tested for elastic deflection as a beam supported at the extreme ends. It bends exactly 1 inch under a central load of 20 lbs. Calculate the breaking weight of the same rod when set upright and used as a column with rounded ends (adopting Euler's formula). *Ans.* 250 lbs.

7. In what cases may Euler's formula be rationally and safely used for calculating the breaking weight of a column? In what other cases would the formula be unreliable, and what kind of error would it introduce?

8. In the case of solid cylindrical columns, what would be your rough estimate of the relative strengths of cast and of wrought iron—(a) when the length of the column is equal to 5 diameters, and (b) when it is 50 diameters? *Ans.* (a) 2·14 : 1; (b) 2·34 : 1.

9. Find the sectional area A, the moment of inertia I (about the diametral axis), and the radius of gyration R, for a hollow cylindrical column whose external diameter is 8 inches, and its internal diameter 6 inches. *Ans.* 22 square inches; 137·5 inches⁴; 2·5 inches.

10. A hollow circular column has a projecting bracket on which a load of 1 ton rests. The centre of this load is 2 feet from the centre of the column. External diameter of column is 10 inches and the thickness 1 inch. What is the maximum compression stress? *Ans.* 1,000 lbs. per square inch.

11. A solid cast-iron column, 6 inches in diameter and 10 feet high, is fixed firmly at each end. What load will it carry when the unit stress is not to exceed 8 tons per square inch? *Ans.* 125 tons.

12. A circular mild-steel tie bar, 3 inches in diameter and 30 feet long, is overloaded until it is fractured. Take 26 tons per square inch as the breaking load, and 10 per cent. as the extension. Limit of elasticity 15 tons per square inch. Calculate by an approximate rule the work done in fracturing the bar. *Ans.* 568 inch-tons.

13. The strut in a framed structure is formed of a steel pipe 6 inches external diameter and ½ inch thick; it is 10 feet long and has pin connections at each end. With a factor of safety of 5, to what load may it be submitted? *Ans.* 30 tons.

14. In which way would a steel column 30 feet long by 9 inches external and 8 inches internal diameter fail, assuming a load to be applied in the direction of the longitudinal neutral axis? Assuming the ends to be hinged, what load would this column safely carry with a factor of safety of 5, the ultimate strength of the material being 30 tons per square inch, and the modulus of elasticity 30,000,000 lbs. per square inch. *Ans.* Rankine, 28·7 ; Euler, 25 tons.

15. A column 20 feet high, fixed in direction at the ends, has to carry a load of 25 tons, and consists of an I section beam, 12 inches deep and 8 inches wide. Find the necessary thickness for the flanges, neglecting the effect of the web and taking a factor of safety of 3. Given that the breaking load $= \dfrac{21\,S}{1 + \dfrac{L^2}{30,000\,k^2}}$, where S = the cross-sectional area, L = the length, and k = the radius of gyration. *Ans.* ·3 inch.

16. A cast-iron strut, 2 feet long, with a sectional area of 8 square inches carries a load of 32 tons. Assuming E = 8,000 tons per square inch, how much is the strut shortened? What is the stress and what is the strain per unit length? *Ans.* ·012 inch ; 4 tons per square inch ; ·0005.

17. A building 20 feet high to the eaves and 50 feet wide is to be constructed of steel columns placed 15 feet apart, with the sides formed of steel framing and plaster slabs. The roof has a rise of 12 feet above the eaves, and its weight is 20 lbs. per square foot. With a wind pressure of 40 lbs. per square foot on the sides and of 2·4 lbs. per square foot normal to the roof, calculate the stresses in the column, and sketch the sections to be adopted ; the maximum stresses not to exceed 5 tons per square inch. Assume the base to be rigidly fixed, and the roof truss to be simply carrying its own weight and not assisting the column.

18. A circular steel chimney 5 feet in diameter outside, is to be built of steel plates ⅜ inch thick ; assuming the diameter of the chimney and thickness of plates to be constant, and the base rigidly fixed to the foundation, how high can the chimney be built to withstand a wind pressure of 40 lbs. per square foot equally distributed over the whole surface, and not to exceed a unit-stress of 6 tons per square inch ?

APPENDIX.

Rules and Syllabus of A.M.Inst.C.E. Examinations, and recent
Examination Papers.

The Centimetre, Gramme, Second, or C.G.S. System of Units of
Measurement and their Definitions — Fundamental Units —
Derived Mechanical Units—Practical Electrical Units.

Properties of British Standard Sections.

Tables of Constants, Logarithms, Antilogarithms, and Functions
of Angles.

The Institution of Civil Engineers.

EXTRACTS FROM RULES AND SYLLABUS OF EXAMINATIONS FOR

ELECTION OF ASSOCIATE MEMBERS.

Note.—Engineers who desire to enter for the A.M.Inst.C.E. examinations should write *at once* to the Secretary, Great George Street, Westminster, S.W., for the complete Rules, Syllabus, and Application Forms.

PART II.*—*Scientific Knowledge.*

SECTION A.

1. Mechanics (one Paper, *time allowed, 3 hours*).

2. Strength and Elasticity of Materials (one Paper, *time allowed, 3 hours*).

3. *Either* (*a*) **Theory of Structures,**
or (*b*) **Theory of Electricity and Magnetism** (one Paper, *time allowed, 3 hours*).

SECTION B.

Two of the following nine subjects—not more than one from any group (one Paper in each subject taken, *time allowed, 3 hours for each Paper*):—

Group i.	*Group* ii.	*Group* iii.
Geodesy.	Hydraulics.	Geology and Mineralogy.
Theory of Heat Engines.	Theory of Machines.	Stability and Resistance of
Thermo- and Electro-Chemistry.	Metallurgy.	Ships.
		Applications of Electricity.

* Candidates may offer themselves for examination in Sections A and B of Part II. together; or they may enter for Section A alone, and, if successful, may take Section B at a subsequent examination. In the latter case, however, such candidates will not be allowed to present themselves for examination in Section B unless or until they are actually occupied in work as pupils or assistants to practising Engineers. The Council may permit Candidates who have attempted the whole of Part II. at one examination, and have failed in Section B only, to complete their qualification by passing in that section at a subsequent examination, subject to their being then occupied as above stated.

Mathematics.—The standard of Mathematics required for the Papers in Part II. of the examination is that of the mathematical portion of the Examination for the Admission of Students, though questions may be set involving the use of higher Mathematics.

The range of the examinations in the several subjects, in each of which a choice of questions will be allowed, is indicated generally hereunder :—

<div align="center">SECTION A.</div>

1. Mechanics :—

Statics.—Forces acting on a rigid body; moments of forces, composition, and resolution of forces ; couples, conditions of equilibrium, with application to loaded structures. The foregoing subjects to be treated both graphically and by aid of algebra and geometry.

Hydrostatics.—Pressure at any point in a gravitating liquid ; centre of pressure on immersed plane areas ; specific gravity.

Kinematics of Plane Motion.—Velocity and acceleration of a point ; instantaneous centre of a moving body.

Kinetics of Plane Motion. — Force, mass, momentum, moment of momentum, work, energy, their relation and their measure ; equations of motion of a particle ; rectilinear motion under the action of gravity ; falling bodies and motion on an inclined plane ; motion in a circle ; centres of mass and moments of inertia ; rotation of a rigid body about a fixed axis ; conservation of energy.

2. Strength and Elasticity of Materials:—

Physical properties and elastic constants of cast iron, wrought iron, steel, timber, stone, and cement ; relation of stress and strain, limit of elasticity, yield-point, Young's modulus ; coefficient of rigidity ; extension and lateral contraction ; resistance within the elastic limit in tension, compression, shear and torsion ; thin shells ; strength and deflection in simple cases of bending; beams of uniform resistance; suddenly applied loads.

Ultimate strength with different modes of loading ; plasticity, working stress ; phenomena in an ordinary tensile test ; stress-strain diagram ; elongation and contraction of area ; effects of hardening, tempering and annealing ; fatigue of metals ; measurement of hardness.

Forms and arrangements of testing machines for tension, compression, torsion, and bending tests; instruments for measuring extension, compression, and twist ; forms of test pieces and arrangements for holding them ; influence of form on strength and elongation ; methods of ordinary commercial testing ; percentage of elongation and contraction of area ; test conditions in specifications for cast iron, mild steel, and cement.

3. (*a*) Theory of Structures :—

Graphic and analytic methods for the calculation of bending moments and of shearing forces, and of the stresses in individual members of framework structures loaded at the joints ; plate and box girders ; incomplete and redundant frames ; stresses suddenly applied, and effects of impact :

buckling of struts; effect of different end fastenings on their resistance; combined strains; calculations connected with statically indeterminate problems, as beams supported at three points, &c.; travelling loads; riveted and pin-joint girders; rigid and hinged arches; strains due to weight of structures; theory of earth - pressure and of foundations; stability of masonry and brickwork structures.

3. (b) Theory of Electricity and Magnetism:—

Electrical and magnetic laws, units, standards, and measurements; electrical and magnetic measuring instruments; the theory of the generation, storage, transformation, and distribution of electrical energy; continuous and alternating currents; arc and incandescent lamps; secondary cells.

Section B.

Group i. Theory of Heat Engines:—

Thermodynamic laws; internal and external work; graphical representation of changes in the condition of a fluid; theory of heat engines working with a perfect gas; air- and gas-engine cycles; reversibility, conditions necessary for maximum possible efficiency in any cycle; properties of steam; the Carnot and Clausius cycles; entropy and entropy-temperature diagrams, and their application in the study of heat engines; actual heat engine cycles and their thermodynamic losses; effects of clearance and throttling; initial condensation; testing of heat engines, and the apparatus employed; performances of typical engines of different classes; efficiency.

Group ii. Hydraulics:—

The laws of the flow of water by orifices, notches, and weirs; laws of fluid friction; steady flow in pipes or channels of uniform section; resistance of valves and bends; general phenomena of flow in rivers; methods of determining the discharge of streams; tidal action; generation and effect of waves; impulse and reaction of jets of water; transmission of energy by fluids; principles of machines acting by weight, pressure, and kinetic energy of water; theory and structure of turbines and pumps.

Theory of Machines:—

Kinematics of machines; inversion of kinematic chains; virtual centres; belt, rope, chain, toothed and screw gearing; velocity, acceleration and effort diagrams; inertia of reciprocating parts; elementary cases of balancing; governors and flywheels; friction and efficiency; strength and proportions of machine parts in simple cases.

Group iii. Applications of Electricity:—

Theory and design of continuous- and alternating-current generators and motors, synchronous and induction motors and static transformers; design

of generating- and sub-stations and the principal plant required in them; the principal systems of distributing electrical energy, including the arrangement of mains and feeders; estimation of losses and of efficiency; principal systems of electric traction; construction and efficiency of the principal types of electric lamps.

☞ Candidates should see, that all their " Forms " are *duly completed* and *passed* by the Council of the Institution of Civil Engineers, Great George Street, Westminster, S.W., *before* 1st January for the February Examination, and *before* the 1st September for the October Examination.

Examinations Abroad.—The papers of the *October* Examination *only* will be placed before accepted Candidates in India and the Colonies.   To enable the Secretary to make arrangements for the Application Forms and Fees, &c., of these Candidates, their Forms, &c., *must* be in the Secretary's hands, *before* the 1st June preceding the October Examinations.

THE INSTITUTION OF CIVIL ENGINEERS' EXAMINATION, FEBRUARY, 1909.

ELECTION OF ASSOCIATE MEMBERS.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

1. State concisely the meaning of the following terms:—Elastic limit, yield point, modulus of elasticity, resilience.

2. A rolled steel joist 12 inches deep has a moment of inertia about the neutral axis parallel to the flanges of 375·6; what is its "section modulus," and what distributed load could it carry when supported upon bearings 10 feet apart if the stress in the extreme fibre did not exceed 8 tons per square inch?

3. A hard steel rod ¾ inch diameter and 20 inches long was shortened by 0·007 inch by a load of 2 tons; what was the stress, the strain, and the modulus of elasticity?

4. Derive from first principles the formula $f = \dfrac{M\,y}{I}$ when $f =$ the unit stress within the elastic limit, M = the bending moment, $y =$ the distance of the extreme fibre from neutral axis, and I = moment of inertia. State any assumption you make.

5. A ball of steel falls 38·82 feet upon a horizontal slab of the same material. Assuming the coefficient of restitution (or resilience) to be 0·7, how high will the ball rise after impact? What is the ratio of kinetic energy just before and just after impact?

6. A steel bar 1 inch square is placed upon bearings 30 inches apart; what load placed in the centre of the bar will deflect it 0·25 inch? Assume the modulus of elasticity to be 30,000,000 lbs. per square inch.

7. Sketch the appearance of the fracture of the following materials when crushed to destruction:—(a) A cube of cast iron, (b) a cube of wrought iron, (c) a cube of sandstone, (d) a piece of pine 2 inches long by 1 inch square placed on end.

8. A steel shaft has to transmit 100 horse-power when running at 300 revolutions per minute; what must be its diameter if the stress in the extreme fibre does not exceed 6 tons per square inch?

9. It is frequently stated that the line of resistance in a masonry structure must not pass outside the "middle third": (a) To what form of section does this apply? (b) What are the assumptions upon which the

statement is based? (c) What is the corresponding limit of deviation in a structure of circular section? Give the proof.

10. Fill in the spaces left in the following table, indicating the properties of the various materials :—

Material.	Breaking Load.		Extension in 8 Inches.	Modulus of Elasticity.	Limits of Elasticity.
	Tensile.	Compressive.			
Steel bars, . .					
Wrought-iron bars,					
Cast iron, . .					
Brass, . . .					
Copper, . . .					
Concrete, . .					
Portland cement, .					
Pitch pine, . .					

11. Sketch a machine for making tensile tests of steel up to 100 tons; mark the leading dimensions.

12. State the advantages and the disadvantages of the use of reinforced concrete for structural purposes, and explain the duty of each material in the compound material.

October, 1909.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

1. A timber beam of 12 feet span is supported at both ends; it is 8 inches deep and 14 inches wide, and carries a brick wall 14 inches thick. Find what height of wall it can carry if the maximum stress in the beam is to be 800 lbs. per square inch, and the weight of brickwork is 125 lbs. per cubic foot.

2. Draw to scale the stress-strain diagram of a tensile test of a steel bar, 1 inch square, and having an ultimate strength of 30 tons per square inch. Explain the meaning of the different parts of the diagram.

3. A load of 3 tons is hung from a round steel rod $\frac{3}{4}$ inch in diameter and 40 feet long. How much will the rod stretch if its modulus of elasticity be 13,000 tons per square inch?

4. A rolled steel joist, 8 inches deep, whose moment of inertia is 55·7 (in inch units), is laid on bearings 10 feet apart, and carries a load of 7 tons uniformly distributed. What is its maximum fibre stress, and what is its central deflection if the modulus of elasticity be 14,000 tons per square inch?

5. In a certain girder a steel flat bar tension member, 10 inches wide and $\frac{1}{2}$ inch thick, carries a load of 30 tons; it is attached to a gusset by a butt joint with two cover plates, rivets of $\frac{3}{4}$ inch diameter being used. Give a dimensioned sketch of the riveting you would propose, and explain fully

your reason for the number of rivets, pitch, lap, and thickness of cover plate which you adopt. Assume the following as limiting stresses, 5·5 tons per square inch in shearing for rivets, 11·0 tons per square inch in bearing for plates, 6·5 tons per square inch in tension for plates.

6. Derive the formula for the torsional resisting moment of a solid cylindrical shaft. What will be the maximum fibre stress of a 3-inch shaft transmitting 100 H.P. at 300 revolutions per minute?

7, Rankine's formula for the crippling load P of a round-ended column of cross-sectional area A is

$$\frac{P}{A} = \frac{f}{1 + a\dfrac{l^2}{k^2}},$$

where l denotes the length of the column and k the radius of gyration of the section, also $a = \dfrac{1}{7,500}$ and $f = 48,000$ lbs. per square inch for mild steel. Use this formula to find the crippling load of a round-ended column made of a 10×8 inch rolled steel joist whose least moment of inertia is 71·6 in inch units, and whose area is 20·6 square inches.

8. A cylindrical steel tank, 30 feet in diameter, is to be filled with water to a depth of 12 feet. What is the thickness of plate and what riveting in the vertical joints would you adopt for the bottom ring of plate? (Weight of water = 62½ lbs. per cubic foot.)

9. What is the least internal radius to which a flat bar ½ inch thick can be bent so that the maximum stress does not exceed 7 tons per square inch, taking the modulus of elasticity at 14,000 tons per square inch?

10. Derive from first principles the moment of resistance of a rectangular beam. (Breadth = b, depth = d.)

11. Describe the effects produced by the several processes of hardening, tempering and annealing upon the physical properties of steel containing various percentages of carbon, from the point of view of the use of the material for engineering purposes.

12. State the physical conditions and phenomena to be determined or observed in the practical examination of Portland cement, and the methods of testing it, giving an outline sketch of some common form of machine for determining its tensile strength.

February, 1910.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

1. State concisely the meaning of the following terms :—Stress ; strain ; modulus of elasticity ; yield point. How is the yield point of steel affected by repeated straining and by annealing?

2. Give a general description of a modern 50-ton testing-machine ; and sketch out an extensometer for use in such a machine.

3. A 12-inch × 12-inch balk of timber spans an opening of 14 feet. What will be the weight per foot run of an evenly distributed load which produces a maximum fibre stress of 700 lbs. per square inch? and how much will the balk deflect under this load if its modulus of elasticity be 1,500,000 lbs. per square inch?

4. A round steel suspension-rod 42 feet 4 inches long and 1⅜ inch diameter is stretched 0·253 inch by a certain load. What is this load if the modulus of elasticity be 28,750,000 lbs. per square inch ?

5. A 10-inch × 6-inch steel bulb tee, whose moment of inertia is 122·278 (in inch units) and centre of gravity is 3·881 inches from the outside face of its table, spans an opening of 14 feet 6 inches. What load must be concentrated at its mid span in order to produce a maximum fibre stress of 6½ tons per square inch ?

6. A steel cylindrcial air-receiver of 67 inches diameter is to carry a pressure of 90 lbs. per square inch ; what thickness of plate would you use for the shell, and what diameter, pitch, and arrangement of rivets in the longitudinal joints, if these are lap joints, double riveted, and if the stresses are limited to 14,500 lbs. per square inch in tension, 12,500 lbs. in shear, and 25,000 per square inch in bearing area ? Sketch out your arrangement of riveting, and state the actual stresses you would have in tension, shear, and bearing.

7. A solid cylindrical steel shaft of 4 inches diameter and running at 200 revolutions per minute transmits 150 H.P. What will be its angle of twist (in degrees) in a length of 46 feet, if the modulus of shear be 11,500,000 lbs. per square inch ?

8. A 10-inch × 6-inch steel-rolled joist used as a strut is subjected to a compression stress of 50 tons acting along an axis lying in the centre line of its web, and 3 inches from the outer face of one flange. Find the maximum fibre stress produced. Moment of inertia = 211·614 (in inch units) and area = 12·358 square inches.

9. Describe the effect produced upon the ultimate strength of steel by repeated stresses (a) when reversed from compression to tension, (b) when varied between two tensions.

10. A steel bolt 1¾ inch diameter is passed through a cast-iron pipe of 2 inches internal and 3 inches external diameter, and 20 inches long, and the nut is screwed up until head and nut bear lightly on the pipe. Find how much more the nut must travel along the bolt in order to put a compression of 4 tons per square inch on the pipe, taking the modulus of elasticity at 29,000,000 lbs. for steel and 14,000,000 lbs. for cast iron.

11. If a cast-iron test bar exactly 2 inches deep, 1 inch wide and 36 inches between points of support breaks with a central load of 3,340 lbs., what should be the breaking weight of another test bar of the same quality but 1·92 inches deep, 0·97 inch wide, and 35·4 inches between points of support ?

12. A wrought-iron rod 50 feet long and ¾ inch diameter has to arrest a weight of 600 lbs. falling through a distance of 6 inches. What will be the maximum stress per square inch imposed thereby on the tie-rod ? Modulus of elasticity = 27,000,000 lbs. per square inch.

October, 1910.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

1. Define " elasticity " and the meaning of Young's modulus E, giving some examples of this modulus in such materials as cast iron, wrought iron, and mild steel. Explain what is meant by the " elastic limit," and describe some of the differences which are to be observed between elastic and plastic strain.

2. Describe some kind of "extensometer" by which it is possible to obtain a fairly accurate measurement of the modulus E in direct tension:— (a) Upon a length of 50 feet, and (b) upon a length of 8 inches ; giving in each case some idea as to the magnitude of the linear dimension that would have to be measured.

3. A straight bar of rectangular section, 6 inches wide and 1 inch deep, is laid across a span of 5 feet, and exhibits a deflection of ⅜ inch under a central load of 10 cwts. Find the modulus E.

4. Construct the diagram of bending moments for the beam described in Question No. 3, giving the maximum bending moment and maximum intensity of fibre stress ; and show how the curve of elastic deflection can be constructed graphically from the diagram of moments.

5. Describe what is meant by the "ultimate elongation" of an 8-inch test-piece (in direct tension) ; how its measured value will be affected by any slight aberration in the prismatic or cylindrical form of the test-piece, and how it enters into the calculated "work of fracture" as indicated upon the load-strain diagram. Discuss also its value (1st) as an indication of the comparative "toughness" of different materials, and (2nd) as a factor in the work of fracture for a long structural member in direct tension.

6. Find the position of the neutral axis in a rolled iron bar of upright **T** section 4 inches wide, 3½ inches deep over all, and ½ inch thick. When this bar is bent (elastically) in the vertical plane, to a radius of 250 feet, calculate the tensile and the compressive stresses in the extreme fibres. (Take $E = 12,000$ tons per square inch.)

7. The chains of a suspension bridge are composed of flat eye-bars of mild steel, 9 inches broad with widened heads and pin-connections, each bar having a uniform thickness of 1 inch. What should be the diameter of the pin and width of head in order to develop the full strength of the bar ? and for what reasons are such dimensions necessary?

8. A steel suspending rod with a sectional area of 1 square inch is carrying a steady load of 6 tons. What will be the momentary stress when (a) the load is suddenly reduced from 6 to 3 tons, and (b) when the load is suddenly increased from 6 to 9 tons?

9. A horizontal beam ABC of uniform section, whose moment of inertia is I, is used as a cantilever anchored down at A upon a solid abutment, and supported at B upon rocker-bearings, while it carries a uniform load w per foot of its entire length. Let l denote the length of each arm AB = BC, and write the expression for the slope of the bent beam at A, at B and at C, and the vertical deflection at C—all referred to the horizontal line ABC. Illustrate by sketch.

10. Find the "radius of gyration" of cross-section for each of the following :—(1) Solid round column, 6 inches diameter ; (2) hollow cylindrical column, 12 inches outside and 10 inches inside diameter ; (3) solid beam, 6 inches square.

11. A bar of cast iron tested in direct tension exhibits an "ultimate strength" $f = 9$ tons per square inch. A bar of the same material is tested as a beam 2 inches deep and 1 inch wide laid upon supports 3 feet apart, and loaded at the centre. Calculate the load W that would suffice to produce the same tensile stress of 9 tons per square inch in the extreme fibre. Will the beam break under that load? or what load will suffice to break it?

12. In making any such calculation as that referred to in question 11, explain the reasoning on which it is based, and the several assumptions that underlie the usual theory of transverse flexure.

February, 1911.

STRENGTH AND ELASTICITY OF MATERIALS.

*Not more than* EIGHT *questions to be attempted by any Candidate.*

1. Give a careful sketch showing the usual form of the " load-strain diagram " for an 8-inch test-piece of mild steel having a sectional area of 1 square inch, with a " yield " at 16·5 tons, an " ultimate strength " of 26 tons, and an " ultimate elongation " of 28 per cent. Refer to the features of the diagram, and describe the process of plastic extension, the local contraction and the ultimate fracture.

2. Taking the specimen in Question 1, and supposing it to remain sensibly cylindrical up to an extension of 20 per cent. under a load of 25·8 tons, find the actual stress at that stage of the process, and go on to trace the " stress-strain " curve as distinguished from the load-strain curve, comparing the two.

3. In what manner is the " ultimate elongation " of a test-piece usually measured and expressed ? How will the measured quantity be affected by the local contraction in test-pieces of different lengths and of different diameters ?

4. Supposing that a tie-bolt of the material described in Question 1 has been turned to a diameter of 1 inch for a length of 3 feet, and that it transmits the pull by a square-threaded screw cut upon it at one end to a depth of $\frac{1}{10}$ inch, reducing the diameter to 0·8 inch inside the thread, how would you then estimate the strength and the ultimate elongation of the tie-bolt ?

5. The floor of a rectangular building, 30 feet long and 20 feet wide, is carried by the two timber beams A C and D F laid across from wall to wall and crossing one over the other at the centre B, where they carry a
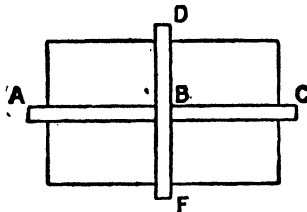
concentrated load of 35 cwts. If the beams are of similar cross-section find how much of the load is carried by each beam, and its greatest bending moment : assuming the modulus E to be the same in both.

6. A solid round bar of mild steel, 2 inches in diameter, is used as a strut 50 inches long with " rounded ends " ; find the crippling load w per square inch :—

(a) By the Rankine formula

$$w = \frac{f}{1 + \dfrac{1}{7,500}\dfrac{l^2}{r^2}},$$

(b) By the Euler formula

$$w = \pi^2 \, E \, . \, \frac{r^2}{l^2}.$$

where l = the length, and r = the radius of gyration; taking π^2 = 10 nearly, E = 29,000,000 lbs., and f = 48,000 lbs. per square inch. Repeat the same calculations for a strut 100 inches in length.

7. Discuss the character and the real meaning of the formulas (*a*) and (*b*) quoted in Question No. 6, explaining their disagreement. In calculating the strength of a strut by either formula how will the result depend upon the freedom or the fixity of its ends ? and how would you treat this question in dealing with the compression members in the boom or the web system of a large girder ?

8. A wrought-iron beam of uniform section 2 inches square is rigidly fixed at both ends in abutments 6 feet apart, and is loaded in the middle of the span with a weight of 1 ton. Sketch the diagram of moments and the curve of deflection, giving the greatest moment and the greatest deflection. (E = 12,000 tons per square inch.)

9. For any beam of uniform section, show how the ordinates of the deflection-curve can be determined by successive integration ; or, as an alternative, how they can be more simply measured in terms of the moment of an area in the bending-moment diagram.

10. A bar of steel, 1 inch square in section and originally straight, is bent under stress to a radius of 1,000 inches ; find the bending moment and the stress per square inch in the extreme fibre. (E = 13,000 tons per square inch.)

11. In the testing of Portland cement under direct tension how is the apparent strength of the briquette affected by the time-rate of loading ?

12. A cross-girder 18 inches deep over all is constructed of steel plate and angle-bars, each flange consisting of a pair of angle-bars $\overline{3\frac{1}{2} \times 3\frac{1}{2} \times \frac{1}{2}}$ riveted to the web-plate 18 inches $\times$ $\frac{1}{2}$ inch, in the usual form. Find the moment of inertia of the cross-section.

October, 1911.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. Define the following terms :—Limit of elasticity, modulus of elasticity, Poisson's ratio, and explain how the value of E is determined experimentally.

2. A steel test bar 1 inch in diameter 8 inches between the gauge points is broken with a tensile load of 25 tons ; draw the probable load-extension diagram of the same, and state what you consider would be the probable percentage of elongation and the contraction of area, the yield point being 17 tons per square inch.

3. A rod 30 feet long having a sectional area of $\frac{3}{4}$ of a square inch suspends a dead load of 3,900 lbs., and has a live load of 900 lbs. suddenly applied without velocity ; assuming E to be 25,000,000 lbs. per square inch, what is the amount of the resilience of the rod when carrying the dead load, and at the instant immediately after the live load comes on ?

4. A load is applied to the crank fixed to a wrought-iron shaft 6 inches diameter and 20 feet long, which twists the ends to the extent of 2° ; assuming the modulus of transverse elasticity (or coefficient of rigidity) to be 4,000 tons per square inch, what is the extreme fibre-stress ?

5. Give the approximate unit crushing loads of the following materials, and make a sketch of the fractures when the test-pieces are crushed :— Cast iron, 2-inch cubes; Aberdeen granite, 3-inch cubes; English oak, 3 inches × 2 inches square, on end ; Portland cement (neat), 2-inch cubes.

6. Specify fully, in abstract form, the necessary properties of the Portland cement used in first-class work, naming the various tests to be made.

7. Three rolled steel joists 6 inches deep are placed side by side spanning an opening of 10 feet ; the moment of inertia of the two outer joists is 20, and that of the inner one 44 inch-units. A central load of 5 tons is so placed as to deflect each of the three joists equally ; state the amount of the load carried by each joist, and the maximum unit-stress in the centre joist only.

8. A piece of flat steel has to be bent round a drum 5 feet in diameter, what is the maximum thickness which the strip can be made so that there shall be no permanent deformation when it is removed from the drum ? The steel has an elastic limit of 14 tons per square inch ; E = 14,000 tons per square inch.

9. Calculate the sectional area, moment of inertia, radius of gyration, and section-modulus for the following section, about the axis A A. State the use of the above properties when calculating the strength of a member of the section shown, submitted to direct and bending stresses either separate
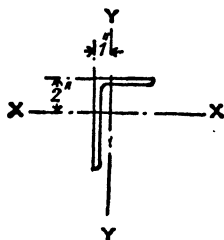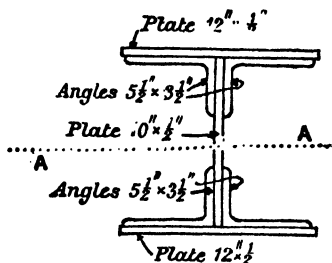
Properties of angle, ... 5½ inches × 3½ inches.
Moment of inertia about Y Y, ... = 3 inch-units.
Sectional area, ... = 3·4 square inches.

or combined. Assume the centre of gravity as sketched in lines of sections X X, Y Y.

10. A vertical pole 40 feet high has a horizontal wire attached to the top, which transmits a pull of 3,000 lbs., the deflection of the pole being 4 inches. What would be the approximate effect on the above deflection if a wire guy, 50 feet long, with a sectional area of $\frac{1}{15}$ of a square inch, were fixed in the same vertical plane as the pole and the horizontal wire, but leading from the top of the pole to the ground, which it strikes at a point 30 feet from the base of the pole ? Assume E = 25,000,000 lbs. per square inch, and neglect the shortening of the post due to the additional load from the wire guy.

11. In designing reinforced-concrete work, what is the usual value to adopt for the ratio of the moduluses of elasticity of steel to that of the concrete, and what is the value of each of the two moduluses ? A vertical reinforced-concrete column has vertical steel rods embedded in the concrete ; if the stress in the concrete is 200 lbs. per square inch, what is the corresponding stress in the steel rods ?

12. In a reinforced-concrete rectangular beam 18 inches deep by 8 inches wide, assume the neutral axis to be 7 inches below the upper surface of the beam, and that the two steel reinforcing bars, having a total area of $1\frac{1}{2}$ square inches, have their centres 1 inch from the bottom of the beam. Find the unit-stress in the rods, and also the maximum unit-stress in the concrete when the section is subjected to a bending moment of 308,000 lb.-inches.

February, 1912.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

In the following questions, E = *modulus of elasticity.*

1. How is the "section-modulus" and radius of gyration of a section of a bar obtained, and how is this applied to ascertaining the strength of a beam? Calculate the section-modulus and radius of gyration of the section, shown by the figure, about the axis Y-Y.

2. A rectangular steel bar 3 inches by 2 inches is placed on edge between bearings 3 feet apart; what load placed at the centre will deflect the bar $\frac{1}{4}$ of an inch? Assume E = 30,000,000 lbs. per square inch.

3. Describe briefly the results of the experiments of Bauschinger, Wöhler, or other experimenters upon the repeated application of a load upon a steel bar. How do these results influence the specification of the various unit-stresses in the design of a structure subjected to a live load?

4. Experiments upon some wrought-iron bars showed that a permanent set was taken when the bars were stressed to a degree greater than that produced by a stress of 20,000 lbs. per square inch, but not when stressed to a less degree. At that point the average strain was 0·0006 foot per foot of length; what was the resilience of this quality of iron, in foot-pounds per square inch section per foot of length?

5. What are the changes which take place in the physical properties of a steel bar when it is drawn into fine wire? To what treatment is the rod submitted before it is drawn into wire?

6. Draw up a concise specification of the physical properties which may be ascertained in a well-equipped testing-house, and by an inspection of the following materials :—Mild steel, cast iron, pitch pine, granite.

7. In designing a compression member, formulæ are made use of which reduce the allowable unit-stress which may be placed in this member, the variation being governed by the ratio of $\dfrac{l}{r} = \dfrac{\text{length of column}}{\text{radius of gyration}}$. Why is this done? Give Euler's theoretical formula for columns, and also some
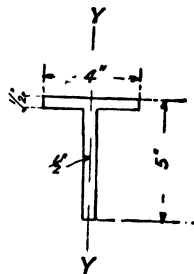
well-known practical formula, and state the approximate limits in terms of $\frac{l}{r}$, within which each of the formulæ should be used in good design.

8. In crane No. 1 a load of 10 tons is suspended from the point A, causing a vertical deflection of 1 inch at the point A. Crane No. 2 is exactly similar to crane No. 1, but, instead of the tie-rod being secured direct to the wall, there is an intervening tension rod, C D, 20 feet long by 1 square inch in section. Assuming this rod to be horizontal, and neglecting the effect of
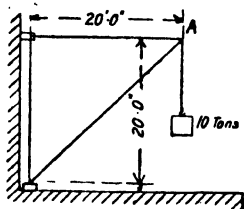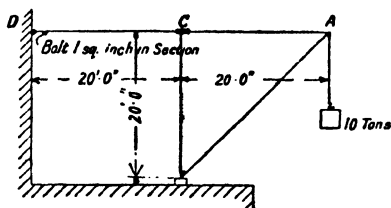
CRANE No. 1. CRANE No. 2.

its own weight, what will be the deflection at the point A in this case caused by the load of 10 tons ? E = 30,000,000 lbs. per square inch.

9. A compression member constructed of concrete is submitted to a load which produces a unit-stress about 50 per cent. in excess of that permissible on the material. Sketch fully, with explanatory notes on sketches, any methods you know for increasing the strength of the member, without altering its over-all dimensions.

10. An iron bar 10 feet long having E = 14,000 tons per square inch, and a limit of elasticity = 14 tons per square inch, is subjected to shocks of a total value of 224 foot-lbs. The bar is not to have any permanent set produced in it, this being guaranteed by the adoption of a factor of safety of 2. Find the required sectional area of the bar.

11. In a thin shell, such as a boiler or riveted pipe, subjected to an internal pressure of p lbs. per square inch, prove that—(a) the total stress per lineal inch upon any part of the shell equals $6\ p\ .\ d$; (b) the unit-stress equals $\frac{6\ p\ .\ d}{t\ e}$.

> d = diameter of shell in feet.
> t = thickness of shell plates in inches.
> e = percentage efficiency of the longitudinal joint.

12. A reinforced-concrete beam of rectangular section 10 inches deep by 5 inches broad carries a load of 3 tons, distributed over a span of 12 feet. The two reinforcing steel rods have a united area of 1·2 square inch, and the centre of them is 1 inch from the bottom of the beam ; find the unit-stresses in the concrete and in the steel rods, at the centre of the beam. Do you consider either of the ascertained stresses excessive ?

October, 1913.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. What is the difference between " stress " and " strain," and between " elastic limit " and " yield-point " ? A steel bar 10 feet long and 2 inches diameter stretches $\frac{1}{32}$ of an inch with a load of 10 tons. What is the modulus of elasticity of the material ?

2. A rectangular steel bar 4 inches × 1 inch is placed on edge on supports 10 feet apart ; if a load of 1 ton be placed in the centre of the bar, how much would it deflect, assuming that no twisting of the bar took place ? Prove the formula you adopt. E = 30,000,000 lbs. per square inch.

3. Describe the properties of Portland cement, hydraulic lime, and Roman cement. State some of the purposes for which each material is well adapted.

4. What is the difference in the temperature coefficient of expansion between steel and concrete ? What would be the effect of submitting reinforced concrete to a temperature of 300° F. ?

5. Draw the stress- (on original area) extension diagram of the testing of (*a*) mild steel bars, (*b*) hard-drawn steel wire, to destruction by a tensile force. Give a comparative description of the two diagrams.

6. A beam carries a uniformly distributed load. Show by sketches, and give an explanation of, the various stresses occurring in the beam, showing lines of principal tensile and compressive stresses and the distribution of shear stresses.

7. What are the advantages and disadvantages of reinforced concrete in structural work, and what is absolutely necessary for its successful adoption ?

8. Describe briefly Wöhler's experiments on repetition of stress, and state the conclusions to which he arrived, and how these affect the design of structures with moving loads.

9. Under the assumptions of " straight line stress variation " and " no tension in concrete " prove the following equation, from which the depth " *n* " of the neutral axis from the extreme compression fibre, for a doubly-reinforced rectangular concrete beam, may be obtained :—

$$n^2 + \frac{2 m (A_c + A_t)}{b} n - \frac{2 m(A_c d_c + A_t d_t)}{b} = 0.$$

m = ratio of E of steel to concrete,
A_t = area of tensile reinforcement,
A_c = area of compressive reinforcement,
d_c = depth of compressive reinforcement from extreme fibre,
d_t = depth of tensile reinforcement from extreme fibre.

Derive the formula giving the value of the lever arm " *a* " of the centres of action of the compressive and tensile forces in the same beam

10. A beam is fixed at one end and freely supported at the other ; it is loaded uniformly over the whole span with " w " pounds per foot run. Find the value of " n " in the following deflection formula :—

$$d = \frac{n\, w\, l^4}{E\, I}.$$

w = load per foot run,
l = span,
I = moment of inertia of cross-section about the neutral axis.

11. How do shear stresses act in a reinforced-concrete beam, and how are they resisted by the reinforcement ? Explain the effect of the reinforcement.

12. Specify the tests for Portland cement as required by the British Standards Committee, enumerating and describing all the necessary accessories for making the tests.

February, 1914.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. Describe briefly, and give an example of each of the following terms :—

      (1) Resilience,
      (2) Poisson's Ratio,
      (3) Modulus of Elasticity.

2. Describe, and illustrate with sketch, a cement-testing machine with automatic appliance for applying the load ; state the necessary precaution to be taken when using it.

3. Derive and prove a formula for ascertaining the stress in a thin shell, such as a boiler. In the case of a boiler, how does the pressure on the end affect the stress in the shell ?

4. How would you test the compressive strength of granite, cast iron, and English oak ? State the approximate strength of each, and sketch the probable form of fracture.

5. Plot a typical stress-extension (on original area) and real stress-extension diagram for the tensile test of a mild-steel bar. Give an explanation of the two diagrams.

6. Assume a uniformly loaded horizontal beam supported at each end ; the stress in the material not exceeding 400 lbs. per square inch, the deflection not exceeding 1 inch in 50 feet of span, the neutral axis to be one-half the depth of the beam and E = 1,200,000 lbs. per square inch. What is the ratio of the depth to the span ?

7. When a steel rod is drawn into fine wire, how are its physical properties affected, and what subsequent operations are necessary to make the wire suitable (a) for the cable of a suspension bridge, (b) for the winding rope of a mine shaft ?

8. What is meant by " seasoning " timber, and how is the timber affected by the operation ? Describe briefly three methods of seasoning timber.

9. Columns of reinforced concrete are reinforced by longitudinal rods usually bound together with (a) straight roads, (b) rings, (c) continuous spirals. Explain the advantages of each system, and state whether the enclosed concrete derives any additional supporting power or not.

10. How are the stresses calculated in a thick cylinder for internal pressure ? Derive a formula to ascertain these stresses, preferably that based upon Barlow's theory.

11. Assuming a uniform rectangular bar to be supported at each end and loaded at the centre, prove that the work done in deflecting the bar within the elastic limit is equal to the volume of the bar multiplied by $\frac{1}{18}\frac{f^2}{E}$, f being the maximum stress on the bar.

12. A mild steel bar is subjected to direct tensile stress of 4 tons per square inch and to a shearing stress of 2 tons per square inch. What is the resultant stress on the material ?

October, 1914.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. Before making a tensile test in a new single-lever testing-machine of the ordinary type, state what observations you would make in order to determine if the scale readings indicated correctly the force on the specimen.

2. Define the terms " yield point " and " elastic limit," and explain briefly how they are measured in a tensile test of mild steel.

3. State the appliances you would require and the method by which you would proceed to determine by a transverse test the value of Young's modulus for mild steel from a piece of rolled joist 6 inches by 3 inches by 7 feet long.

4. Find the diameter of a hollow steel shaft to transmit 120 H.P. at 1,000 revolutions per minute under the following conditions:—Outer radius = 1·5 times inner radius ; maximum stress, 8,000 lbs. per square inch ; and maximum twisting moment assumed equal to twice the mean twisting moment.

5. A hollow cast-iron column of outside diameter 8 inches and thickness ¼ inch supports a load of 2 tons, the line of action of which is parallel to the axis of the column and 2 inches distant from it. Find the maximum and minimum stresses in the metal

6. A ferro-concrete beam has its reinforcement distributed in the manner indicated in the figure. Assuming working stresses of 600 lbs. per square inch for the concrete, and 16,000 lbs. per square inch for the steel, find
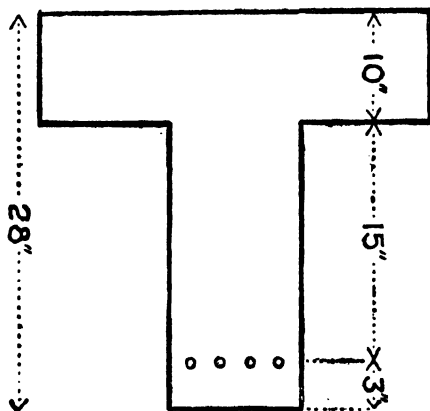
the width of the compression flange and the area of the reinforcement if the beam has to support a uniformly distributed load of 3,000 lbs. per foot run over a span of 20 feet

7. Give a brief description of any testing machine with which you are familiar for determining the tensile strength of cement briquettes, and the method of preparing the briquettes.

8. Assuming an efficiency of 75 per cent. for the longitudinal seams, and a working stress of 10,000 lbs. per square inch, find the thickness of the plates of a cylindrical boiler 7 feet in diameter for a working pressure of 150 lbs. per square inch.

9. State the considerations which would guide you in the choice of a suitable steel for use in high-speed petrol engine crank shafts, and mention any mechanical tests which would give evidence of properties of the material which you consider desirable.

10. The foundation bolts of an engine-bed plate are 30 inches long, 1 inch in diameter, and eight in number. Calculate the maximum energy which they will store up under a limiting stress of 16,000 lbs. per square inch. Diameter at the bottom of the threads = 0·84 inch. E = 30,000,000 lbs. per square inch.

11. Give a short account of any experiments with which you are familiar on the manner in which mild steel fails under combined twisting and tension, and state the conclusions which have been drawn from the results.

12. State Euler's and Rankine's formulæ for the resistance of struts, and discuss the circumstances under which these formulæ may be used for the purposes of design.

February, 1915.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. In making tensile tests of specimens of circular cross-section cut from the same bar of mild steel, state what difference in the results obtained you would expect to find if the ratio of length to diameter varied considerably in different specimens.

2. Define Young's modulus of elasticity and the modulus of torsional elasticity. How does the difference in value of these quantities for a given material affect the work which can be stored in a given volume of it under the respective stress conditions?

3. Find the dimensions of a timber beam of 12 feet span and depth equal to twice the width, to bear a uniformly distributed load of 1,000 lbs. per lineal foot. Maximum stress 1,000 lbs. per square inch.

4. Distinguish between "natural" and artificial limits of elasticity in mild steel, and state the particular treatment by which they may be produced.

5. Describe briefly with sketches any form of testing machine with which you are familiar for making tensile, bending, and compression tests.

6. In determining the tensile strength of cast iron, state what precautions you would take in designing the form of the specimen, so that the true tensile strength may be found. Is the same precaution necessary in mild steel, and under what circumstances?

7. A wall 18 inches wide carries a coping 24 inches wide, which projects 6 inches from the wall on one side. If the weight of the coping is 500 lbs. per lineal foot, find the maximum and minimum intensities of pressure produced by it on the wall.

8. Describe a test for determining the relative hardness of metals and state how the coefficient of hardness is calculated, and if it is an absolute constant for the material.

9. Find the diameter of the wire and the number of coils in a spiral spring which will deflect 2 inches under a load of 50 lbs. Mean diameter of coils 2 inches. Maximum torsional stress 10,000 lbs. per square inch. Modulus of torsional elasticity 10,000,000 lbs. per square inch.

10. Describe how you would reinforce a concrete beam 12 inches wide and 24 inches deep by means of steel rods embedded in the concrete, and find the value of the moment of resistance of the section of the beam, stating carefully the assumptions which you make.

11. A thin hollow shaft is subjected to combined torsion and tension such that the shearing stress on a section perpendicular to the axis is 5 tons per square inch and the tensile stress in the direction of the axis is 10 tons per square inch. Find the maximum stress on the shaft and the plane on which it acts.

12. Calculate the buckling load of a wooden strut 2 inches square in section, 100 inches long, if one end is fixed and the other end free. E for wood = 2,000,000 lbs. per square inch. What modification would you make in the calculation if the strut were 30 inches long, and the known crushing stress of the wood were 4,000 lbs. per square inch?

October, 1915.

## STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. Define the following terms : " stress," " strain," " modulus of elasticity," " resilience." Show the relationship which exists between them.
2. Derive formulas which may be used for ascertaining the longitudinal and circumferential stresses in a thin cylindrical shell.
3. Calculate the deflection at the centre of a timber beam, 12 inches deep and 12 inches wide, having a clear span of 20 feet ; it is loaded centrally so that the maximum fibre stresses do not exceed 1,000 lbs. per square inch. The modulus of elasticity of the timber may be assumed to be 1,000,000 lbs. per square inch.
4. A hollow shaft is 9 inches external diameter and 4 inches internal diameter. Compare the strength of this shaft with that of a solid shaft of the same weight per foot run, under the two following conditions :—

> (*a*) To resist torsional stresses.
> (*b*) To resist bending stresses.

(Do not consider the case when (*a*) and (*b*) can exist simultaneously.)
5. A reinforced concrete pile is used to resist a lateral pressure ; its dimensions are as follows :—
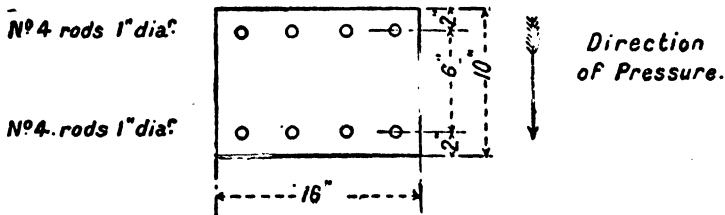
The unit stresses are not to exceed 600 and 16,000 lbs. per square inch respectively in the concrete and the steel ; find the maximum value of the moment of resistance to comply with these requirements.
6. Assuming that the stress over a cross-section varies uniformly, derive a formula which will permit of the calculation of the intensity of stress at any point of a section, when it is submitted to the action of a direct force and a bending moment. A rolled steel joist, 5 feet long, 6 inches wide, rests upon a wall 2 feet thick and projects over the edge of the wall 3 feet ; at the projecting end it is loaded with a weight of 1 ton ; equilibrium is maintained by a load of 2¾ tons placed at the opposite end of the joist, which is in line with the back of the wall. Find the intensities of pressure between the joist and the wall. Neglect the weight of the joist and assume that it is perfectly rigid.

7. State the value of the modulus of elasticity of mild steel and also of concrete; show any variations which occur by varying the proportion of the composition of the concrete. Show exactly how these two properties enter into the consideration of stresses in reinforced concrete.

8. Describe, and illustrate with sketches, the instruments used in an ordinary works testing-house for obtaining the various necessary particulars of a specimen about to be submitted to a tensile test. Consider the original and final cross-sections, gauge points, ultimate extension, contraction of area at fracture, and any other particulars you may consider necessary.

9. State the tests required for Portland cement by the Engineering Standards Committee; make sketches, with any necessary explanatory notes, of the instruments required to carry out the tests.

10. Draw two curves on the same diagram to show:

(a) Load-extension curve,
(b) Stress-extension curve,

for a specimen of mild steel submitted to a tensile test. Mark on your diagram the principal points of stress variation, enlarge any portions of the diagram which may be necessary; show the scales to which it is drawn.

11. Columns may be classed, generally, as follows:—

(a) Fixed at one end, free at the other.
(b) Both ends pivoted.
(c) Fixed at one end, pivoted at the other.
(d) Fixed at both ends.

Assume columns of the same length and form; state how fixing the ends as above detailed would affect the crippling load of the columns. Give the proportion of the varying strengths. Sketch the manner in which the bending will take place.

12. A flitch beam is formed of two timbers 9 inches by 3 inches, and a steel plate 9 inches by ½ inch, all assumed to be efficiently bolted together so as to develop their full strength. The modulus of elasticity of the timber is 1,000,000 lbs. per square inch, and that of the steel 30,000,000 lbs. per square inch. Find the maximum safe value of the moment of resistance so that the unit stresses may not exceed either 1,000 lbs. per square inch in the timber or 15,000 lbs. per square inch in the steel.

February, 1916.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*
In the following questions E *denotes the modulus of elasticity.*

1. Explain carefully the following types of stress, and give examples of their occurrence both separately and in combination: "shear," "tension," "compression," and "torsion."

2. A tie-rod having a sectional area of 2 square inches is 30 feet long;

it is extended 0·2 inch by the action of a certain force; find the value of this force, also the stress, strain, and resilience of the bar when thus extended. E = 30,000,000 lbs. per square inch.

3. A thin spherical shell is submitted to the action of an internal pressure; derive a formula by means of which the stress in the shell may be obtained. The internal pressure on such a shell, 20 inches diameter, is 700 lbs. per square inch; what is the stress in the metal of the shell if it is ¼ inch thick ?

4. A shaft, 3 inches diameter, is connected to its next length by means of a flat-faced flange coupling, the two portions of which are bolted together with five bolts arranged in a pitch circle 7 inches diameter. The shaft transmits 50 H.P. when running at 300 revolutions per minute. Find the diameter of the bolts required in the connection; the unit stress in the bolts is not to exceed the maximum stress in the shaft.

5. Derive the theoretical formula, known as Euler's, for the strength of struts; state any approximations or assumptions adopted in the derivation. Is this formula practically useful ? If so, explain by means of a diagram its limitations in actual practice. Why are other formulæ utilised for finding the strength of a strut within certain limits? Sketch a curve to show the actual crippling strength of a steel column having rounded ends, plot these values to one axis of the diagram and to the other plot the values of the ratio l/r from zero to 200, where l denotes the length of the column and r its minimum radius of gyration. Indicate the portion, if any, of the curve which is derived from Euler's formula.

6. What are the tests required for wrought iron of good quality, such as that known as Grade A in the specification of the Engineering Standards Committee ? Pay particular attention to those tests which are carried out without the aid of a testing machine.

7. Derive a formula for obtaining the deflection of a beam of uniform section submitted to the action of a central load. Find the deflection of a cast-iron beam, 2 inches wide and 3 inches deep, having a clear span of 2 feet 6 inches, when submitted to a central load of 1 ton. The modulus of elasticity of cast iron is 18,000,000 lbs. per square inch.

8. Make sketches to show the mechanical arrangements of a testing machine which can be used for either tensile or compressive specimens; assume that loads up to 50 tons can be applied.

9. A concrete beam 12 inches wide by 18 inches deep is to be reinforced, to resist tensile stresses, with steel rods placed with their centres 2 inches from the surface of the concrete. The unit stresses in the steel and concrete are to be 16,000 lbs. and 600 lbs. per square inch respectively. What is the sectional area of the required reinforcement and the moment of resistance of the beam when fully stressed ?

10. State in the form of a table the values of the modulus of elasticity, yield point, elastic limit, ultimate strength, and extension per cent. in a length of 8 inches, all of these particulars being assumed to have been obtained from tensile tests of the following materials:—Portland cement, oak, cast iron, mild steel, and hard drawn steel wire. Where values are omitted in your table, state your reason very concisely.

11. Make careful sketches of the usual type of tensile test piece that would be made from a mild steel flat bar, the whole section of which is 4 inches by ¼ inch. The first sketch should show the test piece before testing; the second the portions of the broken test piece brought together for measurements to be taken. Fully dimension both sketches and write upon them the information that would be required for the preparation of a works test sheet. Assume values that you would expect to find in the

above test and make out a full test sheet showing all particulars and the results of the test.

12. In making calculations for the design of reinforced-concrete work, what are the assumptions that are generally made with regard to the elasticity and strength of the steel and concrete ? Explain the effect of these assumptions ; do you consider that they are borne out in ordinary practice ? Give your reasons.

February, 1917.

STRENGTH AND ELASTICITY OF MATERIALS.

Not more than EIGHT *questions to be attempted by any Candidate.*

1. Define the coefficients of elasticity and rigidity, and explain by means of a sketch the difference between the " elastic limit " and the " yield point " of a steel specimen tested in tension.

2. Indicate by a sketch how cubes of Portland cement mortar would be likely to fail under direct pressure, and explain the nature of the stresses that would come into play in causing the rupture.

3. Describe mechanism by means of which a testing machine can be caused to draw the stress-strain diagram of a specimen under test.

4. Draw approximately the stress-strain diagram for a round mild-steel bar 0·8 inch diameter and 10 inches long ruptured in tension, if the extension before breaking occurs is 25 per cent. of its original length, and the modulus of elasticity is 13,000 tons per square inch. Show how to calculate the work done in breaking the bar.

5. A wrought-iron bar is 30 inches long and has a cross-sectional area of 0·93 square inch ; if the extension under a load of 8 tons is 0·021 inch, find its modulus of elasticity.

6. Mention the chief points that it is desirable to specify with respect to the quality of the metal for large cast-iron pipes.

7. Explain the nature of the stress that comes into play when a spiral spring is extended. The diameter of the wire forming a spiral spring of 29 coils is 0·25 inch ; the outside diameter of the spring is 2¾ inches. If a weight of 50 lbs. is hung on the spring, calculate the extension of the spring, given that

$$\text{the extension} = \frac{2 \text{ W } a^2 l}{\pi \text{ C } r^4},$$

where W is the weight applied in lbs., C is the modulus of rigidity = 5,000 tons per square inch, a, l, and r are respectively the mean radius of the coil, the length of wire in the spring and the radius of the wire, all in inches.

8. A steel member is 12 feet long and has its ends fixed so that it is unable to alter in length. If the member is subject to a range of temperature of 40° F. above or below the mean, and the coefficient of expansion of the steel is 0·0000066 per degree Fahr., find the maximum stress induced in the member due to the range of temperature above or below its mean

value, given that the modulus of elasticity of the steel is 13,000 tons per square inch.

9. A steel water-pipe 24 inches diameter has to resist the pressure due to a head of 400 feet of water; what thickness should it be made if the working intensity of pressure in the metal is to be 5 tons per square inch after the pipe has lost $\frac{1}{10}$ inch of its thickness, due to corrosion?

10. Define "modulus of section." If the bending moment at the centre of a beam symmetrical about its neutral axis and uniformly loaded is 26 tons feet, the moment of inertia of the cross-section there is 315 inches⁴, and the depth between the centres of the flanges 12 inches, what is the intensity of stress in the flanges at this section?

11. A steel girder 40 feet long carries a uniformly distributed load. If the deflection is not to exceed $\frac{1}{1000}$th part of its length, find the maximum ratio of span to depth at the centre that can be permitted if f the stress in the flange at the centre is to be limited to 6 tons, given that

$$\text{The deflection at the centre in feet} = \frac{5}{24} \times \frac{L^2 \times f}{D \times E},$$

where L and D are respectively the length and depth at centre of the girder in feet, and E is the modulus of elasticity = 13,000 tons per square inch.

12. What are the chief tests that Portland cement is required to stand by the British standard specification?

APPENDIX D.

THE CENTIMETRE, GRAMME, SECOND, OR C.G.S. SYSTEM OF
    UNITS OF MEASUREMENT AND THEIR DEFINITIONS.*

I. Fundamental Units.—The C.G.S. and the practical electrical
units are derived from the following mechanical units :—
    The *Centimetre* as a unit of *length ;* the *Gramme* as a unit of *mass ;* and
the *Second* as a unit of *time*.
    The *Centimetre* (cm.) is equal to 0·3937 inch in length, and nominally
represents one thousand-millionth part, or $\frac{1}{1,000,000,000}$, of a quadrant
of the earth.
    The *Gramme* (gm.) is equal to 15·432 grains, and represents the mass of
a cubic centimetre of water at 4° C. Also, 1 lb. of 16 ozs. is equal to 453·6
grammes. *Mass* (M) is the quantity of matter in a body.
    The *Second* (*s*) is the time of one swing of a pendulum making 86,164·09
swings in a sidereal day, or the $\frac{1}{86,400}$ part of a mean solar day.

II. Derived Mechanical Units.—
    Area (A or cm.²).—The unit of area is the *square centimetre*.
    Volume (V or cm.³).—The unit of volume is the *cubic centimetre*.
    Velocity (*v* or cm./s) is rate of change of position. It involves the idea
of direction as well as that of magnitude. *Velocity* is *uniform* when equal
distances are traversed in equal intervals of time. The unit of velocity is
the velocity of a body which moves through unit distance in unit time, or
the *velocity of one centimetre per second*.
    Momentum (M *v*, or gm. × cm./s) is the quantity of motion in a body, and
is measured by mass × velocity.
    Acceleration (*a* or cm./s²) is the rate of change of velocity, whether that
change takes place in the direction of motion or not. The unit of accelera-
tion is the acceleration of a body which undergoes unit change of velocity
in unit time, or an acceleration of one centimetre per second per second.
The acceleration due to gravity is considerably greater than this, for the
velocity imparted by gravity to falling bodies in one second is about 981
centimetres per second (or about 32·2 feet per second). The value differs
slightly in different latitudes. At Greenwich the value of the acccleration
due to gravity is $g = 981·17$; at the Equator, $g = 978·1$; and at the North
Pole, $g = 983·1$.

* The Author is indebted to his Publishers, Charles Griffin & Co., for
liberty to abstract the following pages on this subject from the latest
edition of Munro and Jamieson's *Pocket-Book of Electrical Rules and
Tables for Engineers and Electricians*, to which the student is referred for
further values and definitions.—A. J.

Force (**F** or *f*) is that which tends to alter a body's natural state of rest or uniform motion in a straight line.

Force is measured by the rate of change of momentum which it produces, or mass × acceleration.

The *Unit of Force*, or *Dyne*, is that force which, acting for one second on a mass of one gramme, gives to it a velocity of one centimetre per second. The force with which the earth attracts any mass is usually called the "weight" of that mass, and its value obviously differs at different points of the earth's surface. The force with which a body gravitates—*i.e.*, its weight (in dynes), is found by multiplying its mass (in grammes) by the value of *g* at the particular place where the force is exerted.

Work is the product of a force and the distance through which it acts. The unit of work is the work done in overcoming unit force through unit distance—*i.e.*, in pushing a body through a distance of one centimetre a force of one dyne. It is called the *Erg*. Since the "weight" of 1 gramme is 1 × 981 or 981 dynes, the work of raising 1 gramme through the height of 1 centimetre against the force of gravity is 981 ergs or *g* ergs. One kilogramme-metre = 100,000 (*g*) ergs. One foot-pound = 13,825 (*g*) ergs = 1·356 × 10⁷ ergs.

Energy is that property which, possessed by a body, gives it the capability of doing work. *Kinetic* energy is the work a body can do in virtue of its motion. *Potential* energy is the work a body can do in virtue of its position. The unit of energy is the *Erg*.

Power or *Activity* is the rate of working. The unit is called the *Watt* (**W**) = 10⁷ ergs per second, or the work done at the rate of 1 *Joule* (**J**) per second.

One Horse-power (**H.P.**) = 33,000 ft.-lbs. per minute = 550 ft.-lbs. per second; but, as seen above under *Work*, 1 ft.-lb. = 1·350 × 10⁷ ergs, and, under *Power*, 1 *Watt* = 10⁷ ergs per second.

Hence, a *Horse-power* = 550 × 1·356 × 10⁷ ergs per sec. = **746 watts**.

If E = volts, C = amperes, and R = ohms,

then,
$$\text{H.P.} = \frac{E\,C}{746} = \frac{C^2\,R}{746} = \frac{E^2}{746\,R}.$$

PRACTICAL ELECTRICAL UNITS.

1. As a **Unit of Resistance** (R), the International **Ohm** (O or ω), which is based upon the ohm equal to 10^9 units of resistance of the C.G.S. system of electro-magnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross-sectional area and of the length of 106·3 centimetres.

2. As a **Unit of Current** (C or c), the International **Ampere** (A), which is one-tenth of the unit of current of the C.G.S. system of electro-magnetic units, and which is represented sufficiently well for practical use by the unvarying current which, when passed through a solution of nitrate of silver in water, and in accordance with their specifications, deposits silver at the rate of 0·001118 gramme per second.

3. As a **Unit of Electro-motive Force** (E), the International **Volt** (V), which is the E.M.F. that, steadily applied to a conductor whose resistance is one International Ohm, will produce a current of one International Ampere, and which is represented sufficiently well for practical use by $\frac{1000}{1434}$ of the E.M.F. between the poles or electrodes of the voltaic cell known as Clark's cell, at a temperature of 15° Centigrade, and prepared in the manner described in their specification, or by the *new* Weston cell

4. As the **Unit of Quantity** (Q), the International **Coulomb** (A × s), which is the quantity of electricity transferred by a current of one International Ampere in one second.

5. As the **Unit of Capacity** (K), the International **Farad** (Fd), which is the capacity of a conductor charged to a potential of one International Volt by one International Coulomb of electricity.

6. As a **Unit of Work** the **Joule** (J), or **Watt-second** (W_P × s), which is 10^7 units of work in the C.G.S. system, and which is represented sufficiently well for practical use by the energy expended in one second *in heating* an International Ohm.

7. As the **Unit of Power** (P_W), the International **Watt** (W_P), which is equal to 10^7 units of power in the C.G.S. system, and which is represented sufficiently well for practical use by the work done at the rate of one Joule per second. The **Kilowatt** (Kw.) = 1,000 Watts = $1\frac{1}{3}$ Horse-power.
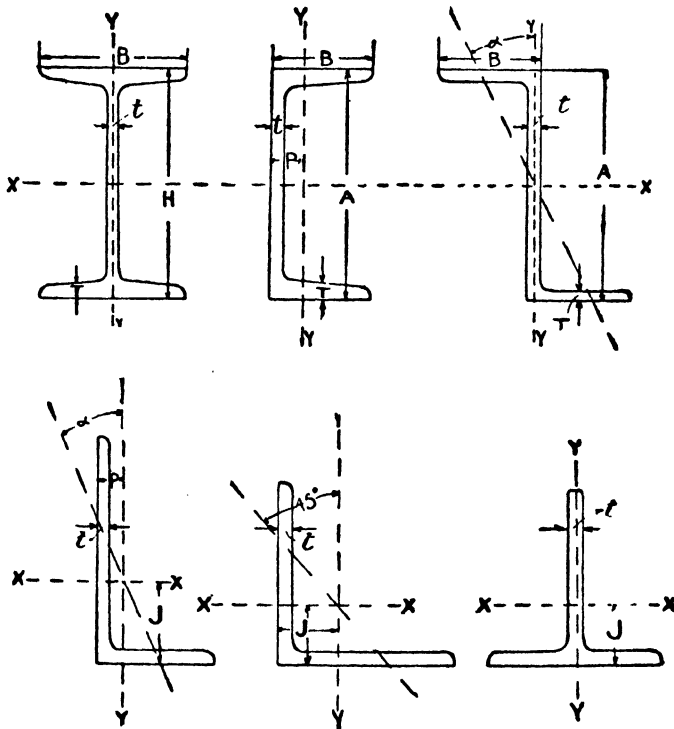
8. As the **Unit of Induction** (L), the **Henry** (H), which is the induction in the circuit when the E.M.F. induced in this circuit is one International Volt while the inducing current varies at the rate of one ampere per second.

9. The **Board of Trade Commercial Unit of Work** (B.T.U.) is the **Kilowatt-hour** (Kw·hr.) = 1,000 Watt-hours = $1\frac{1}{3}$ H.P. working for *one* hour. Or, say, 10 amperes flowing in a circuit for 1 hour at a pressure of 100 volts.

NOTE.—For further simple explanations, with examples, see the latest edition of Prof. Jamieson's *Manual of Practical Magnetism and Electricity.* Also, see the latest edition of Munro and Jamieson's *Electrical Engineering Pocket-Book*—both published by Charles Griffin & Co., London.

PROPERTIES OF BRITISH STANDARD SECTIONS.

The following Tables gives the properties of the British Standard Sections which are usually listed by makers.

BRITISH STANDARD SECTIONS.* (See Fig.)

PROPERTIES OF BRITISH STANDARD ⊥ BEAMS.

Size. H.	B.	Wt. per Foot.	Thickness. t.	T.	Sectional Area.	Moments of Inertia. X X	Y Y	Section Moduli. X X	Y Y	Radii of Gyration. X X.	Y Y.
Inches.		Lb.	Ins.	Ins.	Sq. Ins	Ins.	Ins.	Ins.	Ins.	Ins.	Ins
3 ×	1½	4	·16	·248	1·18	1·66	·124	1·11	·165	1·19	·325
3 ×	3	8½	·20	·332	2·50	3·79	1·16	2·53	·841	1·23	·710
4 ×	1¾	5	·17	·240	1·47	3·67	·194	1·84	·222	1·58	·363
4 ×	3	9¼	·22	·336	2·80	7·53	1·28	3·76	·854	1·64	·677
4½ ×	1¾	6¼	·18	·323	1·91	6·77	·263	2·85	·300	1·88	·371
5 ×	3	11	·22	·376	3·24	13·6	1·46	5·45	·974	2·05	·672
5 ×	4½	18	·29	·448	5·29	22·7	5·66	9·08	2·51	2·07	1·03
6 ×	3	12	·26	·348	3·53	20·2	1·34	6·74	·892	2·40	·616
6 ×	4½	20	·37	·431	5·88	34·7	5·41	11·6	2·40	2·43	·959
6 ×	5	25	·41	·520	7·35	43·6	9·11	14·5	3·64	2·44	1·11
7 ×	4	16	·25	·387	4·71	39·2	3·41	11·2	1·71	2·89	·851
8 ×	4	18	·28	·402	5·30	55·7	3·57	13·9	1·79	3·24	·821
8 ×	5	28	·35	·575	8·24	89·4	10·3	22·3	4·10	3·29	1·12
8 ×	6	35	·44	·597	10·3	111	17·9	27·6	5·98	3·28	1·32
9 ×	4	21	·30	·460	6·18	81·1	4·20	18·0	2·10	3·62	·824
9 ×	7	58	·55	·924	17·1	230	46·3	51·1	13·2	3·67	1·65
10 ×	5	30	·36	·552	8·82	146	9·78	29·1	3·91	4·06	1·05
10 ×	6	42	·40	·736	12·4	212	22·9	42·3	7·64	4·14	1·36
10 ×	8	70	·60	·970	20·6	345	71·6	69·0	17·9	4·09	1·87
12 ×	5	32	·35	·550	9·41	220	9·74	36·7	3·90	4·84	1·02
12 ×	6	44	·40	·717	12·9	315	22·3	52·6	7·42	4·94	1·31
12 ×	6	54	·50	·883	15·9	376	28·3	62·6	9·43	4·86	1·33
14 ×	6	46	·40	·698	13·5	441	21·6	62·9	7·20	5·71	1·26
14 ×	6	57	·50	·873	16·8	553	27·9	76·2	9·31	5·64	1·29
15 ×	5	42	·42	·647	12·4	428	11·9	57·1	4·78	5·89	·983
15 ×	6	59	·50	·880	17·3	629	28·2	83·9	9·40	6·02	1·28
16 ×	6	62	·55	·847	18·2	726	27·1	90·7	9·02	6·31	1·22
18 ×	7	75	·55	·928	22·1	1150	46·6	128	13·3	7·22	1·45
20 ×	7½	89	·60	1·01	26·2	1671	62·6	167	16·7	7·99	1·55
24 ×	7½	100	·60	1·07	29·4	2655	66·9	221	17·8	9·50	1·51

* Published by permission of the Engineering Standards Committee.   The Tables of British Standard ⊥ Beams, Channels, and Zed Bars are reprinted from Report No. 6 as issued by the Committee   Additional calculations have been inserted in the Tables of British Standards Unequal Angles, Equal Angles, and Tee Bars for thicknesses other than those calculated by the Committee, such calculations having been taken by permission from the *Pocket Companion* issued by Messrs. Dorman, Long & Co., Ltd.

PROPERTIES OF BRITISH STANDARD CHANNELS.

Size. A × B.	Standard Thicknesses.		Wt. per Foot.	Area.	Dimension P.	Moments of Inertia.		Section Moduli.		Radii of Gyration.	
	t.	T.				About XX.	About YY.	About XX.	About YY.	About XX.	About YY.
Ins.	Ins.	Ins.	Lbs.	Sq. Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.
15 × 4	·525	·630	41·94	12·334	·935	377·0	14·55	50·27	4·748	5·53	1·09
12 × 4	·525	·625	36·47	10·727	1·031	218·2	13·65	36·36	4·599	4·51	1·13
12 × 3½	·500	·600	32·88	9·671	·867	190·7	8·922	31·79	3·389	4·44	·960
12 × 3½	·375	·500	26·10	7·675	·860	158·6	7·572	26·44	2·868	4·55	·993
11 × 3½	·475	·575	29·82	8·771	·896	148·6	8·421	27·02	3·234	4·12	·980
10 × 4	·475	·575	30·16	8·871	1·102	130·7	12·02	26·14	4·147	3·84	1·16
10 × 3½	·475	·575	28·21	8·296	·933	117·9	8·194	23·59	3·192	3·77	·994
10 × 3½	·375	·500	23·55	6·925	·933	102·6	7·187	20·52	2·800	3·85	1·02
9 × 3½	·450	·550	25·39	7·469	·971	88·07	7·660	19·57	3·029	3·43	1·01
9 × 3½	·375	·500	22·27	6·550	·976	79·90	6·963	17·76	2·759	3·49	1·03
9 × 3	·375	·437	19·37	5·696	·754	65·18	4·021	14·48	1·790	3·38	·840
8 × 3½	·425	·525	22·72	6·682	1·011	63·76	7·067	15·94	2·839	3·09	1·03
8 × 3	·375	·500	19·30	5·675	·844	53·43	4·329	13·36	2·008	3·07	·873
7 × 3½	·400	·500	20·23	5·950	1·061	44·55	6·498	12·73	2·664	2·74	1·04
7 × 3	·375	·475	17·56	5·166	·874	37·63	4·017	10·75	1·889	2·70	·882
6 × 3½	·375	·475	17·9	5·266	1·119	29·66	5·907	9·885	2·481	2·36	1·06
6 × 3	·312	·437	14·49	4·261	·938	24·01	3·503	8·003	1·699	2·37	·907

PROPERTIES OF BRITISH STANDARD ZED BARS.

Size. A × B.	Standard Thicknesses.		Area.	Wt. per Foot.	Moments of Inertia.		Section Moduli.		Angle a in degrees.	Least Radius of Gyration.
	t.	T.			About XX.	About YY.	About XX.	About YY.		
Ins.	Ins.	Ins.	Sq. Ins	Lbs.	Ins.	Ins.	Ins.	Ins.		Ins.
10 × 3½	·475	·575	8·283	28·16	117·865	12·876	23·573	3·947	14	·839
9 × 3½	·450	·550	7·449	25·33	87·889	12·418	19·531	3·792	16½	·843
8 × 3½	·425	·525	6·670	22·68	63·729	12·024	15·932	3·657	19½	·845
7 × 3½	·400	·500	5·948	20·22	44·609	11·618	12·745	3·521	23	·840
6 × 3½	·375	·475	5·258	17·88	29·660	11·134	9·887	3·361	28½	·821
5 × 3	·350	·450	4·169	14·17	16·145	6·578	6·458	2·328	29½	·698

PROPERTIES OF BRITISH STANDARD UNEQUAL ANGLES.

Size and Thickness.	Area.	Wt. per Foot.	Dimensions.		Moments of Inertia.		Section Moduli.		Angle a in degrees.	Least Radius of Gyration.
			J.	P.	About X X.	About Y Y.	About X X.	About Y Y.		
Ins. 7 × 3½ × ½	Sq. Ins 5·0	Lbs. 17·00	Ins. 2·50	Ins. ·764	Ins. 25·1	Ins. 4·28	Ins. 5·58	Ins. 1·56	14½	Ins. ·74
7 × 3½ × ⅝	6·172	20·98	2·55	·814	30·55	5·15	6·86	1·92	14½	·74
7 × 3½ × ¾	7·313	24·86	2·60	·862	35·68	5·95	8·11	2·26	14	·73
6½ × 4½ × ½	5·248	17·84	2·08	1·09	22·2	8·75	5·02	2·57	25	·97
6½ × 4½ × ⅝	6·482	22·04	2·13	1·14	27·09	10·60	6·20	3·15	25	·96
6½ × 4½ × ¾	7·686	26·13	2·18	1·19	31·66	12·32	7·33	3·72	25	·96
6½ × 3½ × ⅜	3·610	12·27	2·22	·741	15·7	3·27	3·67	1·18	16½	·75
6½ × 3½ × ½	4·750	16·15	2·28	·792	20·4	4·20	4·83	1·55	16½	·75
6½ × 3½ × ⅝	5·860	19·92	2·33	·841	24·83	5·06	5·95	1·90	16	·74
6 × 4 × ⅜	3·610	12·27	1·91	·923	13·2	4·73	3·23	1·54	23½	·87
6 × 4 × ½	4·750	16·15	1·96	·974	17·1	6·10	4·23	2·02	23½	·86
6 × 4 × ⅝	5·860	19·92	2·02	1·02	20·8	7·36	5·23	2·47	23½	·86
6 × 3½ × ⅜	3·424	11·64	2·01	·773	12·6	3·22	3·16	1·18	19	·76
6 × 3½ × ½	4·502	15·31	2·06	·823	16·4	4·14	4·16	1·55	19	·75
6 × 3½ × ⅝	5·549	18·87	2·11	·872	19·88	4·97	5·11	1·89	18½	·75
5½ × 3½ × ⅜	3·236	11·00	1·80	·807	9·93	3·15	2·68	1·17	22	·76
5½ × 3½ × ½	4·252	14·46	1·85	·857	12·80	4·05	3·51	1·53	22	·75
5½ × 3½ × ⅝	5·236	17·80	1·90	·905	15·6	4·86	4·33	1·87	21½	·75
5½ × 3 × ⅝	3·050	10·37	1·90	·662	9·45	2·02	2·62	·86	17	·64
5½ × 3 × ½	4·003	13·61	1·95	·711	12·2	2·58	3·44	1·13	16½	·64
5½ × 3 × ⅝	4·925	16·74	2·00	·759	14·7	3·08	4·20	1·37	16½	·63
5 × 4 × ⅜	3·236	11·00	1·51	1·01	7·96	4·53	2·28	1·52	32	·85
5 × 4 × ½	4·252	14·46	1·56	1·06	10·3	5·82	2·99	1·98	32	·84
5 × 4 × ⅝	5·236	17·80	1·60	1·11	12·4	7·01	3·66	2·43	32	·83
5 × 3½ × ⅜	3·050	10·37	1·59	·848	7·64	3·09	2·24	1·17	25½	·75
5 × 3½ × ½	4·003	13·61	1·64	·897	9·86	3·96	2·93	1·52	25½	·75
5 × 3½ × ⅝	4·925	16·74	1·69	·944	11·9	4·75	3·60	1·86	25	·74
5 × 3 × ₁₆	2·402	8·17	1·66	·667	6·14	1·68	1·84	·72	20	·65
5 × 3 × ⅜	2·859	9·72	1·68	·693	7·24	1·97	2·18	·85	19½	·65
5 × 3 × ½	3·749	12·75	1·73	·742	9·33	2·51	2·85	1·11	19½	·64
5 × 3 × ⅝	4·609	15·67	1·78	·789	11·25	3·00	3·49	1·36	19	·64

UNEQUAL ANGLES (*continued*).

Size and Thickness.	Area.	Wt. per Foot.	Dimensions.		Moments of Inertia.		Section Moduli.		Angle a in degrees.	Least Radius of Gyration.
			J.	P.	About XX.	About YY.	About XX.	About YY.		
Ins.	Sq. Ins	Lbs.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.		Ins
4½×3½×ᵣ₆	2·402	8·17	1·36	·866	4·22	2·55	1·54	·97	30¼	·74
4½×3½× ⅜	2·859	9·72	1·39	·891	5·69	3·00	1·83	1·15	30¼	·74
4½×3½× ½	3·749	12·75	1·44	·940	7·31	3·84	2·39	1·5	30	·74
4½×3½× ⅝	4·609	15·67	1·48	·987	8·81	4·61	2·92	1·83	30	·74
4 ×3½×ᵣ₆	2·246	7·64	1·16	·915	3·46	2·47	1·22	·96	37	·72
4 ×3½× ⅜	2·671	9·08	1·19	·941	4·08	2·90	1·45	1·13	37	·72
4 ×3½× ½	3·499	11·90	1·24	·990	5·23	3·71	1·89	1·48	37	·71
4 ×3½× ⅝	4·295	14·61	1·28	1·04	6·28	4·44	2·31	1·80	36½	·71
4 ×3 ×ᵣ₆	2·091	7·11	1·24	·746	3·31	1·59	1·20	·71	28½	·64
4 ×3 × ⅜	2·485	8·45	1·27	·771	3·89	1·87	1·42	·84	28½	·64
4 ×3 × ½	3·251	11·05	1·31	·819	4·98	2·37	1·85	1·09	28½	·63
4 ×3 × ⅝	3·985	13·55	1·36	·865	5·96	2·83	2·26	1·33	28	·63
3½×3 ×ᵣ₆	1·934	6·58	1·04	·792	2·27	1·53	·92	·69	35½	·62
3½×3 × ⅜	2·298	7·81	1·07	·819	2·67	1·80	1·10	·83	35½	·62
3½×3 × ½	3·001	10·20	1·11	·867	3·40	2·28	1·42	1·07	35½	·61
3½×3 × ⅝	3·673	12·49	1·16	·912	4·05	2·71	1·73	1·30	35	·61
3½×2½×ᵣ₆	1·799	6·05	1·12	·627	2·15	·910	·90	·49	26½	·54
3½×2½× ⅜	2·111	7·18	1·15	·652	2·52	1·06	1·07	·57	26	·53
3½×2½× ½	2·752	9·36	1·20	·699	3·20	1·34	1·39	·74	26	·53
3 ×2½× ¼	1·312	4·46	·895	·648	1·14	·716	·54	·39	34	·52
3 ×2½× ⅜	1·921	6·53	·945	·697	1·62	1·02	·79	·57	34	·52
3 ×2½× ½	2·499	8·50	·992	·744	2·05	1·28	1·02	·73	33½	·52
3 ×2 × ¼	1·187	4·04	·976	·482	1·06	·373	·52	·25	23½	·43
3 ×2 × ⅜	1·733	5·89	1·03	·532	1·50	·525	·76	·36	23	·42
3 ×2 × ½	2·249	7·65	1·07	·578	1·89	·656	·98	·46	22½	·42
2½×2 × ¼	1·063	3·61	·774	·527	·636	·359	·37	·24	32	·42
2½×2 ×ᵣ₆	1·309	4·45	·799	·552	·770	·433	·45	·30	31½	·42
2½×2 × ⅜	1·547	5·26	·823	·575	·895	· 502	·53	·35	31½	·42
2 ×1½×ᵣ₆	·622	2·11	·627	·381	·240	·115	·17	·10	28½	·32
2 ×1½× ¼	·814	2·77	·653	·407	·308	·146	·23	·13	28	·31
2 ×1½×ᵣ₆	·997	3·39	·678	·431	·369	·174	·28	·16	28	·31

BRITISH STANDARD EQUAL ANGLES.

Sizes.	Area.	Weight per Foot.	J.	I_{xx}.	Section Modulus about X X.	Least Radius of Gyration.
Inches.	Sq. Inches.	Lbs.	Inches.	Inches.	Inches.	Inches.
8 × 8 × ½	7·75	26·35	2·15	47·4	8·10	1·58
8 × 8 × ⅝	9·61	32·67	2·20	58·2	10·03	1·57
8 × 8 × ¾	11·44	38·89	2·25	68·5	11·91	1·56
6 × 6 × 7⁄16	5·06	17·21	1·64	17·3	3·97	1·18
6 × 6 × ⅝	7·11	24·18	1·71	23·8	5·55	1·18
6 × 6 × ¾	8·44	28·70	1·76	27·8	6·56	1·17
5 × 5 × ⅜	3·61	12·27	1·37	8·51	2·24	·98
5 × 5 × ½	4·75	16·15	1·42	11·0	3·07	·98
5 × 5 × ⅝	5·86	19·92	1·47	13·4	3·80	·98
4½ × 4½ × ⅜	3·24	11·00	1·22	6·14	1·87	·88
4½ × 4½ × ½	4·25	14·46	1·29	7·92	2·47	·87
4½ × 4½ × ⅝	5·24	17·80	1·34	9·56	3·03	·87
4 × 4 × ⅜	2·86	9·72	1·12	4·26	1·48	·78
4 × 4 × ½	3·75	12·75	1·17	5·46	1·93	·77
4 × 4 × ⅝	4·61	15·67	1·22	6·56	2·36	·77
3½ × 3½ × 5⁄16	2·09	7·11	·97	2·39	·95	·68
3½ × 3½ × ⅜	2·48	8·45	1·00	2·80	1·12	·68
3½ × 3½ × ½	3·25	11·05	1·05	3·57	1·46	·68
3½ × 3½ × ⅝	3·98	13·55	1·00	4·27	1·77	·68
3 × 3 × ¼	1·44	4·90	·827	1·21	·56	·59
3 × 3 × ⅜	2·11	7·18	·877	1·72	·81	·58
3 × 3 × ½	2·75	9·36	·924	2·19	1·05	·58
3 × 3 × ⅝	3·36	11·43	·970	2·59	1·28	·58
2½ × 2½ × ¼	1·19	4·04	·703	·677	·38	·48
2½ × 2½ × 5⁄16	1·46	4·08	·728	·822	·46	·48
2½ × 2½ × ⅜	1·73	5·89	·752	·962	·55	·48
2½ × 2½ × ½	2·25	7·65	·799	1·21	·71	·48
2¼ × 2¼ × 3⁄16	·809	2·75	·616	·378	·23	·44
2¼ × 2¼ × ¼	1·06	3·61	·643	·489	·30	·44
2¼ × 2¼ × 5⁄16	1·31	4·45	·668	·592	·37	·43
2¼ × 2¼ × ⅜	1·55	5·26	·692	·686	·44	·43
2 × 2 × 3⁄16	·715	2·43	·554	·260	·18	·39
2 × 2 × ¼	·938	3·19	·581	·336	·24	·39
2 × 2 × 5⁄16	1·15	3·92	·605	·401	·29	·38
2 × 2 × ⅜	1·36	4·62	·629	·467	·34	·38
1¾ × 1¾ × 3⁄16	·622	2·11	·495	·172	·14	·34
1¾ × 1¾ × ¼	·814	2·77	·520	·220	·18	·34
1¾ × 1¾ × 5⁄16	·997	3·39	·544	·264	·22	·34
1½ × 1½ × 3⁄16	·526	1·79	·434	·105	·10	·29
1½ × 1½ × ¼	·686	2·33	·458	·134	·13	·29
1½ × 1½ × 5⁄16	·839	2·85	·482	·159	·16	·29
1¼ × 1¼ × 3⁄16	·433	1·47	·371	·058	·07	·24
1¼ × 1¼ × ¼	·561	1·91	·396	·073	·09	·23

BRITISH STANDARD TEES.

Sizes.	Area.	Weight per Foot.	J.	Moments of Inertia.		Section Moduli.		Radii of Gyration.	
				X X.	Y Y.	X X.	Y Y.	X X.	Y Y.
Ins.	Sq. Ins.	Lbs.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.
6 ×4 × ⅜	3·634	12·36	·915	4·70	6·34	1·52	2·11	1·14	1·32
6 ×4 × ½	4·771	16·22	·968	6·07	8·62	2·00	2·87	1·13	1·34
6 ×4 × ⅝	5·878	19·99	1·02	7·35	10·91	2·47	3·64	1·12	1·36
6 ×3 × ⅜	3·260	11·08	·633	2·06	6·39	·87	2·13	·795	1·40
6 ×3 × ½	4·272	14·53	·684	2·63	8·65	1·14	2·88	·785	1·42
6 ×3 × ⅝	5·256	17·87	·732	3·14	10·94	1·39	3·65	·773	1·44
5 ×4 × ⅜	3·257	11·07	·998	4·47	3·69	1·49	1·48	1·17	1·06
5 ×4 × ½	4·268	14·51	1·05	5·77	5·02	1·96	2·01	1·16	1·08
5 ×3 ×	2·875	9·78	·691	1·97	3·71	·85	1·49	·828	1·14
5 ×3 × ½	3·762	12·79	·741	2·52	5·03	1·11	2·01	·818	1·16
4 ×4 ×	2·872	9·77	1·11	4·19	1·90	1·45	·95	1·21	·814
4 ×4 × ½	3·758	12·78	1·16	5·40	2·59	1·90	1·29	1·20	·830
4 ×3 ×	2·498	8·49	·767	1·86	1·91	·83	·96	·863	·875
4 ×3 × ½	3·262	11·08	·816	2·36	2·60	1·08	1·30	·851	·893
3½×3½ ×	2·496	8·49	·98	2·79	1·28	1·10	·73	1·05	·717
3½×3½ × ½	3·259	11·08	1·04	3·54	1·75	1·44	1·00	1·04	·733
3 ×3 ×	2·121	7·21	·868	1·70	·816	·80	·54	·897	·620
3 ×3 × ½	2·760	9·38	·918	2·16	1·11	1·04	·74	·886	·636
3 ×2½ ×	1·929	6·56	·695	1·01	·814	·56	·45	·725	·650
3 ×2½ × ½	2·506	8·52	·742	1·28	1·28	·73	·74	·713	·665
2½×2½ × ¼	1·197	4·07	·697	·677	·302	·38	·24	·752	·502
2½×2½ × ₅⁄₁₆	1·474	5·01	·724	·832	·387	·46	·31	·747	·512
2½×2½ × ⅜	1·742	5·92	·750	·959	·473	·55	·38	·742	·521
2½×2½ × ¼	1·071	3·64	·638	·488	·224	·30	·20	·675	·457
2½×2½ × ⅜	1·554	5·28	·689	·685	·349	·44	·31	·664	·474
2 ×2 × ¼	·947	3·22	·579	·337	·157	·24	·16	·597	·407
2 ×2 × ⅜	1·367	4·64	·628	·469	·246	·34	·25	·586	·424
1½×2 × ¼	·820	2·79	·648	·307	·068	·23	·09	·612	·288
1½×2 × ₅⁄₁₆	1·003	3·41	·674	·369	·088	·28	·12	·607	·296
1½×1½ × ¼	·820	2·79	·519	·221	·107	·18	·12	·520	·361
1½×1¾ ×	·999	3·40	·544	·265	·137	·22	·16	·515	·370
1½×1½ × ₅⁄₁₆	·531	1·81	·435	·106	·048	·10	·06	·447	·301
1½×1½ × ¼	·692	2·35	·460	·135	·067	·13	·09	·442	·312

EXAMINATION TABLES.

USEFUL CONSTANTS.

1 Inch = 25·4 millimetres.

1 Gallon = ·1605 cubic foot = 10 lbs. of water at 62° F.   ∴ 1 lb. = ·01605 cubic foot.

1 Knot = 6080 feet per hour.   1 Naut = 6080 feet.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 Cubic foot of water weighs 62·3 lbs.

1 Cubic foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.

1 Cubic foot of Hydrogen at 0° C. and 1 atmosphere, weighs ·00557 lb.

1 Foot-pound = $1·3562 \times 10^7$ ergs.

1 Horse-power-hour = 33000×60 foot-pounds.

1 Electrical unit = 1000 watt-hours.

Joule's Equivalent to suit Regnault's H, is $\begin{cases} 774 \text{ ft.-lbs.} = 1 \text{ Fah. unit.} \\ 1393 \text{ ft.-lbs.} = 1 \text{ Cent. } \text{„} \end{cases}$

1 Horse-power = 33000 foot-pounds per minute = 746 watts.

Volts × amperes = watts.

1 Atmosphere = 14·7 lb. per square inch = 2116 lbs. per square foot = 760 m.m. of mercury = 10^6 dynes per sq. cm. nearly.

A Column of water 2·3 feet high corresponds to a pressure of 1 lb. per square inch.

Absolute temp., $t = \theta° C. + 273°·7$.

Regnault's H = $606·5 + ·305 \, \theta° C. = 1082 + ·305 \, \theta° F$.

$p \, u^{1·0646} = 479$

$\log_{10} p = 6·1007 - \dfrac{B}{t} - \dfrac{C}{t^2}$

where $\log_{10} B = 3·1812$, $\log_{10} C = 5·0871$,

p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

$\pi = 3·1416 = \dfrac{22}{7} = \dfrac{355}{113} = 10(\sqrt{3} - \sqrt{2})$.

One radian = 57·3 degrees.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithm is $e = 2·7183$.

The value of g at London = 32·182 feet per second per second.

TABLE OF LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE OF LOGARITHMS.—*Continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	6	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	5
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE OF ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 2
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 2
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

TABLE OF ANTILOGARITHMS.—*Continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

2   18

TABLE OF FUNCTIONS OF ANGLES.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	000	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1768	5·6713	·9848	1·286	1·3963	80
11	·1929	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
								Radians.	Degrees.
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Angle.	

INDEX FOR VOLUME II.

STRENGTH AND ELASTICITY OF MATERIALS.